Hyperuniform States of Matter: Overview and Progress Report

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HYPERUNIFORMITY

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Hyperuniformity provides a unified means of categorizing and characterizing crystals, quasicrystals and special disordered systems.

Disordered hyperuniform many-particle systems can be regarded to be new ideal states of disordered matter in that they

1. behave more like crystals or quasicrystals in the manner in which they suppress large-scale density fluctuations, and yet are also like liquids and glasses since they are statistically isotropic structures with no Bragg peaks;
2. can exist as both as equilibrium and nonequilibrium phases;
3. come in quantum-mechanical and classical varieties;
4. and, appear to be endowed with unique bulk physical properties.

Understanding such disordered states of matter requires new theoretical tools and present experimental challenges.
Curiosities Circa 2002

- **Coulombic systems** (Martin & Yalcin 1980; Lebowitz 1983): charge fluctuations within some finite-sized window grow like the window surface area.

- **Disordered jammed packings**

  Structure factor $S(k)$ appears to vanish in the limit $k \to 0$: very unusual behavior for a disordered system.

- **Peebles-Harrison-Zeldovich spectrum** for density fluctuations in the early Universe: $S(k) \sim k$ for sufficiently small $k$.

  Gabrielli, Joyce & Sylos Labini 2002
Local Density Fluctuations for General Point Patterns

Torquato and Stillinger, PRE (2003)

Points can represent molecules of a material, stars in a galaxy, or trees in a forest. Let \( \Omega \) represent a \textit{spherical} window of radius \( R \) in \( d \)-dimensional Euclidean space \( \mathbb{R}^d \).

Average number of points in window of volume \( v_1(R) \): \( \langle N(R) \rangle = \rho v_1(R) \sim R^d \)

Local number variance: \( \sigma^2(R) \equiv \langle N^2(R) \rangle - \langle N(R) \rangle^2 \)
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Local number variance: $\sigma^2(R) \equiv \langle N^2(R) \rangle - \langle N(R) \rangle^2$

For Poisson point patterns and many disordered point patterns, $\sigma^2(R) \sim R^d$.

We call point patterns whose variance grows more slowly than $R^d$ (window volume) hyperuniform. This implies that structure factor $S(k) \to 0$ for $k \to 0$.

All perfect crystals and many perfect quasicrystals are hyperuniform such that $\sigma^2(R) \sim R^{d-1}$: number variance grows like window surface area.

SCATTERING AND DENSITY FLUCTUATIONS
Pair Statistics in Direct and Fourier Spaces

For particle systems in $\mathbb{R}^d$ at number density $\rho$, $g_2(r)$ is a nonnegative radial function that is proportional to the probability density of pair distances $r$.

The nonnegative structure factor $S(k) \equiv 1 + \rho \tilde{h}(k)$ is obtained from the Fourier transform of $h(r) = g_2(r) - 1$, which we denote by $\tilde{h}(k)$.

Poisson Distribution (Ideal Gas)

Liquid

Lattice
Hidden Order on Large Length Scales

Which is the hyperuniform pattern?
Scaled Number Variance for 3D Systems at Unit Density

\[ \sigma^2(R)/R^3 \]

- disordered non-hyperuniform
- ordered hyperuniform
- disordered hyperuniform

R

0 4 8 12 16 20
Remarks About Equilibrium Systems

For single-component systems in equilibrium at average number density $\rho$,

$$\rho k_B T \kappa_T = \frac{\langle N^2 \rangle_* - \langle N \rangle^2_*}{\langle N \rangle_*} = S(k = 0) = 1 + \rho \int_{\mathbb{R}^d} h(r) \, dr$$

where $\langle \rangle_*$ denotes an average in the grand canonical ensemble.

Some observations:

- Any ground state ($T = 0$) in which the isothermal compressibility $\kappa_T$ is bounded and positive must be hyperuniform. This includes crystal ground states as well as exotic disordered ground states, described later.

- However, in order to have a hyperuniform system at positive $T$, the isothermal compressibility must be zero; i.e., the system must be incompressible.

- Note that generally $\rho k T \kappa_T \neq S(k = 0)$.

$$X = \frac{S(k = 0)}{\rho k_B T \kappa_T} - 1 : \text{ Nonequilibrium index}$$
ENSEMBLE-AVERAGE FORMULATION

For a translationally invariant point process at number density $\rho$ in $\mathbb{R}^d$:

$$\sigma^2(R) = \langle N(R) \rangle \left[ 1 + \rho \int_{\mathbb{R}^d} h(r) \alpha_2(r; R) \, dr \right]$$

$\alpha_2(r; R)$ - scaled intersection volume of 2 windows
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$\alpha_2(r; R)$- scaled intersection volume of 2 windows

For a class of systems and large $R$, we can show

$$\sigma^2(R) = 2^d \phi \left[ A \left( \frac{R}{D} \right)^d + B \left( \frac{R}{D} \right)^{d-1} + o \left( \frac{R}{D} \right)^{d-1} \right],$$

where $A$ and $B$ are the “volume” and “surface-area” coefficients:

$$A = S(k = 0) = 1 + \rho \int_{\mathbb{R}^d} h(r) \, dr, \quad B = -c(d) \int_{\mathbb{R}^d} h(r) r \, dr,$$

$D$: microscopic length scale, $\phi$: dimensionless density

**Hyperuniform**: $A = 0$, $B > 0 \implies$ Sum rule: $\rho \int_{\mathbb{R}^d} h(r) \, dr = -1$

**Hyposurfical**: $A > 0$, $B = 0$

We’ll see that you can have other variance scalings between $R^{d-1}$ and $R^d$. 
\( h(r) \) can be divided into direct correlations, via function \( c(r) \), and indirect correlations:

\[
\tilde{c}(k) = \frac{\tilde{h}(k)}{1 + \rho \tilde{h}(k)}
\]
INVERTED CRITICAL PHENOMENA: Ornstein-Zernike Formalism

$h(r)$ can be divided into direct correlations, via function $c(r)$, and indirect correlations:

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For any hyperuniform system, $\tilde{h}(k = 0) = -1/\rho$, and thus $\tilde{c}(k = 0) = -\infty$. Therefore, at the “critical” reduced density $\phi_c$, $h(r)$ is short-ranged and $c(r)$ is long-ranged.

This is the inverse of the behavior at liquid-gas (or magnetic) critical points, where $h(r)$ is long-ranged (compressibility or susceptibility diverges) and $c(r)$ is short-ranged.
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- For sufficiently large $d$ at a disordered hyperuniform state, whether achieved via a nonequilibrium or an equilibrium route,

$$c(r) \sim -\frac{1}{r^{d-2+\eta}} \quad (r \to \infty), \quad \tilde{c}(k) \sim -\frac{1}{k^{2-\eta}} \quad (k \to 0),$$

$$h(r) \sim -\frac{1}{r^{d+2-\eta}} \quad (r \to \infty), \quad S(k) \sim k^{2-\eta} \quad (k \to 0),$$

where $(2 - d) < \eta < 2$ is a new critical exponent.

- One can think of a hyperuniform system as one resulting from an effective pair potential $v(r)$ at large $r$ that is a generalized Coulombic interaction between like charges. Why? Because

$$\frac{v(r)}{k_B T} \sim -c(r) \sim \frac{1}{r^{d-2+\eta}} \quad (r \to \infty)$$

- However, long-range interactions are not required to drive a nonequilibrium system to a disordered hyperuniform state.
We showed

\[ \sigma^2(R) = 2^d \phi \left( \frac{R}{D} \right)^d \left[ 1 - 2^d \phi \left( \frac{R}{D} \right)^d + \frac{1}{N} \sum_{i \neq j} \alpha_2(r_{ij}; R) \right] \]

where \( \alpha_2(r; R) \) can be viewed as a repulsive pair potential:
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Finding global minimum of \( \sigma^2(R) \) equivalent to finding ground state.
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For large \( R \), in the special case of hyperuniform systems,

\[ \sigma^2(R) = \Lambda(R) \left( \frac{R}{D} \right)^{d-1} + \mathcal{O} \left( \frac{R}{D} \right)^{d-3} \]
Averaging fluctuating quantity $\Lambda(R)$ gives coefficient of interest:

$$\overline{\Lambda} = \lim_{L \to \infty} \frac{1}{L} \int_{0}^{L} \Lambda(R) dR$$
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We showed that for a lattice

$$\sigma^2(R) = \sum_{q \neq 0} \left( \frac{2\pi R}{q} \right)^d [J_{d/2}(qR)]^2, \quad \overline{\Lambda} = 2^d \pi^{d-1} \sum_{q \neq 0} \frac{1}{|q|^{d+1}}.$$ 

Epstein zeta function for a lattice is defined by

$$Z(s) = \sum_{q \neq 0} \frac{1}{|q|^{2s}}, \quad \text{Re } s > d/2.$$ 

Summand can be viewed as an inverse power-law potential. For lattices, 
minimizer of $Z(d + 1)$ is the lattice dual to the minimizer of $\overline{\Lambda}$.

Sarnak and Strömbergsson (2006)

Surface-area coefficient $\overline{\Lambda}$ provides useful way to rank order crystals, quasicrystals and special correlated disordered point patterns.
Averaging fluctuating quantity $\Lambda(R)$ gives coefficient of interest:

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Sarnak and Strömbergsson (2006)

Surface-area coefficient $\overline{\Lambda}$ provides useful way to rank order crystals, quasicrystals and special correlated disordered point patterns.

For certain $d$, minimizer of Epstein zeta function is related to the optimal sphere packing. Henry Cohn will be speaking about the latter topic.
The surface-area coefficient $\overline{\Lambda}$ for some crystal, quasicrystal and disordered one-dimensional hyperuniform point patterns.

<table>
<thead>
<tr>
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<th>$\overline{\Lambda}$</th>
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<tr>
<td>Integer Lattice</td>
<td>$1/6 \approx 0.166667$</td>
</tr>
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<td>Step+Delta-Function $g_2$</td>
<td>$3/16 = 0.1875$</td>
</tr>
<tr>
<td>Fibonacci Chain*</td>
<td>0.2011</td>
</tr>
<tr>
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<td>$1/4 = 0.25$</td>
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More recent work on hyperuniformity of quasicrystals: Oguz, Socolar, Steinhardt and Torquato (2016).
Quantifying Suppression of Density Fluctuations at Large Scales: 2D

The surface-area coefficient $\bar{\Lambda}$ for some crystal, quasicrystal and disordered two-dimensional hyperuniform point patterns.

<table>
<thead>
<tr>
<th>2D Pattern</th>
<th>$\bar{\Lambda}/\phi^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular Lattice</td>
<td>0.508347</td>
</tr>
<tr>
<td>Square Lattice</td>
<td>0.516401</td>
</tr>
<tr>
<td>Honeycomb Lattice</td>
<td>0.567026</td>
</tr>
<tr>
<td>Kagomé Lattice</td>
<td>0.586990</td>
</tr>
<tr>
<td><strong>Penrose Tiling</strong></td>
<td>0.597798</td>
</tr>
<tr>
<td>Step+Delta-Function $g_2$</td>
<td>0.600211</td>
</tr>
<tr>
<td>Step-Function $g_2$</td>
<td>0.848826</td>
</tr>
</tbody>
</table>

*Zachary & Torquato (2009)*
Quantifying Suppression of Density Fluctuations at Large Scales: 3D

Contrary to conjecture that lattices associated with the densest sphere packings have smallest variance regardless of \( d \), we have shown that for \( d = 3 \), BCC has a smaller variance than FCC.

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<tr>
<td>BCC Lattice</td>
<td>1.24476</td>
</tr>
<tr>
<td>FCC Lattice</td>
<td>1.24552</td>
</tr>
<tr>
<td>HCP Lattice</td>
<td>1.24569</td>
</tr>
<tr>
<td>SC Lattice</td>
<td>1.28920</td>
</tr>
<tr>
<td>Diamond Lattice</td>
<td>1.41892</td>
</tr>
<tr>
<td>Wurtzite Lattice</td>
<td>1.42184</td>
</tr>
<tr>
<td>Damped-Oscillating ( g_2 )</td>
<td>1.44837</td>
</tr>
<tr>
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<td>1.52686</td>
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Carried out analogous calculations in high \( d \) (Zachary & Torquato, 2009), of importance in communications. Disordered point patterns may win in high \( d \) (Torquato & Stillinger, 2006).
An interesting 1D hyperuniform point pattern is the distribution of the nontrivial zeros of the Riemann zeta function (eigenvalues of random Hermitian matrices and bus arrivals in Cuernavaca): Dyson, 1962, 1970; Montgomery, 1973; Krbálek & Šeba, 2000; \( g_2(r) = 1 - \sin^2(\pi r)/(\pi r)^2 \)

1D point process is always negatively correlated, i.e., \( g_2(r) \leq 1 \) and pairs of points tend to repel one another, i.e., \( g_2(r) \to 0 \) as \( r \) tends to zero.
1D Translationally Invariant Hyperuniform Systems

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Dyson mapped this problem to a 1D log Coulomb gas at positive temperature: \( k_B T = 1/2 \). The total potential energy of the system is given by

\[
\Phi_N(r^N) = \frac{1}{2} \sum_{i=1}^{N} |r_i|^2 - \sum_{i \leq j} \ln(|r_i - r_j|).
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\]

Constructing and/or identifying homogeneous, isotropic hyperuniform patterns for \( d \geq 2 \) is more challenging. We now know of many more examples.
More Recent Examples of Disordered Hyperuniform Systems

Physical Examples

- **Disordered classical ground states**: Uche et al. PRE (2004)

- **Fermionic point processes**: \( S(k) \sim k \) as \( k \to 0 \) (ground states and/or positive temperature equilibrium states): Torquato et al. J. Stat. Mech. (2008); Scardicchio et al., PRE, 2009

- **Maximally random jammed (MRJ) particle packings**: \( S(k) \sim k \) as \( k \to 0 \) (nonequilibrium states): Donev et al. PRL (2005); Jiao et al. (2011).

- **Ultracold atoms** (nonequilibrium states): Lesanovsky et al. PRE (2014)


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Natural Disordered Hyperuniform Systems

- **Immune-system receptors** (nonequilibrium states): Balasubramanian et al. PNAS (2015)
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- Ultracold atoms (nonequilibrium states): Lesanovsky et al. PRE (2014)
- Random organization (nonequilibrium states): Chaikin et al. (2013); Hexner et al. PRL (2015); Jack et al. PRL (2015); Dreyfus et. al. PRL (2015); Tjhung et al. PRL (2015)

Natural Disordered Hyperuniform Systems


Nearly Hyperuniform Disordered Systems

- Amorphous Silicon (nonequilibrium states): Henja et al. PRB (2013)
- Structural Glasses (nonequilibrium states): Marcotte et al. (2013)
Hyperuniformity and Spin-Polarized Free Fermions

One can map random Hermitian matrices (GUE), fermionic gases, and zeros of the Riemann zeta function to a unique hyperuniform point process on $\mathbb{R}$. 
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\[ g_2(r) = 1 - \frac{2\Gamma(1 + d/2) \cos^2 \left( r K - \frac{\pi(d + 1)}{4} \right)}{K^{\frac{d}{2}+1} r^{d+1}} \quad (r \to \infty) \]

\[ S(k) = \frac{c(d)}{2K} k + O(k^3) \quad (k \to 0) \quad (K : \text{Fermi sphere radius}) \]
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One-Component Plasma (OCP): Ginibre (1965) Ensemble

A hyperuniform determinantal point process is generated by 2D OCP: particles of charge \( e \) interacting via the Coulomb potential immersed in a rigid, uniform background of opposite charge.


For a special coupling constant \( \Gamma = e^2/k_B T \) equal to 2, Jancovici (Phys. Rev. Lett, 1981):

\[
h(r) = -\exp (-\pi r^2) \quad \Rightarrow \quad \text{Long-range pair correlations are not required!}
\]

\[
S(k) = 1 - \exp[-k^2/(4\pi)] \quad \Rightarrow \quad S(k) \sim k^2 \quad (k \to 0)
\]
Hyperuniformity and Jammed Packings

Conjecture: All homogeneous, strictly jammed saturated sphere packings are hyperuniform (Torquato & Stillinger, 2003).
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A 3D maximally random jammed (MRJ) packing is a prototypical glass in that it is maximally disordered but perfectly rigid (infinite elastic moduli).

Such packings of identical spheres are hyperuniform \( S(k) \sim k \) for small \( k \) with quasi-long-range (QLR) pair correlations in which \( h(r) \) decays as \( -1/r^4 \) (Donev, Stillinger & Torquato, PRL, 2005).

Contrast with hard-sphere fluids with correlations that decay exponentially fast.
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Apparently, hyperuniform QLR correlations with decay \(-1/r^{d+1}\) are a universal feature of general MRJ packings in \( \mathbb{R}^d \).

Zachary, Jiao and Torquato, PRL (2011): ellipsoids, superballs, sphere mixtures

Berthier et al., PRL (2011); Kurita and Weeks, PRE (2011): sphere mixtures

Jiao and Torquato, PRE (2011): polyhedra
**In the Eye of a Chicken: Photoreceptors**

- **Optimal** spatial sampling of light requires that photoreceptors be arranged in the **triangular lattice** (e.g., insects and some fish).

- **Birds** are highly **visual** animals, yet their cone photoreceptor patterns are **irregular**.
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**5 Cone Types**

[Jiao, Corbo & Torquato, PRE (2014).]
Avian Cone Photoreceptors

Disordered mosaics of both total population and individual cone types are effectively hyperuniform, which has been never observed in any system before (biological or not). We term this multi-hyperuniformity.

Jiao, Corbo & Torquato, PRE (2014)
Slow and Rapid Cooling of a Liquid

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Slow and Rapid Cooling of a Liquid

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- Typically, ground states are periodic with high crystallographic symmetries.

- Can classical ground states derived from nontrivial interactions ever be disordered?
Collective-Coordinate Simulations

Consider a system of \( N \) particles with configuration \( r^N \) in a fundamental region \( \Omega \) under periodic boundary conditions) with a pair potentials \( v(r) \) that is bounded with Fourier transform \( \tilde{v}(k) \).
Collective-Coordinate Simulations

- Consider a system of $N$ particles with configuration $r^N$ in a fundamental region $\Omega$ under periodic boundary conditions) with a pair potentials $v(r)$ that is **bounded** with Fourier transform $\tilde{v}(k)$.

  The total energy is

  $$\Phi_N(r^N) = \sum_{i<j} v(r_{ij})$$

  $$= \frac{N}{2|\Omega|} \sum_k \tilde{v}(k) S(k) + \text{constant}$$

- For $\tilde{v}(k)$ **positive** $\forall \ 0 \leq |k| \leq K$ and zero otherwise, finding configurations in which $S(k)$ is constrained to be zero where $\tilde{v}(k)$ has support is equivalent to globally **minimizing** $\Phi(r^N)$.

These **hyperuniform** ground states are called “stealthy” and generally highly **degenerate**.
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For \( \tilde{v}(\mathbf{k}) \) positive \( 0 \leq |\mathbf{k}| \leq K \) and zero otherwise, finding configurations in which \( S(\mathbf{k}) \) is constrained to be zero where \( \tilde{v}(\mathbf{k}) \) has support is equivalent to globally minimizing \( \Phi(\mathbf{r}^N) \).

These hyperuniform ground states are called “stealthy” and generally highly degenerate.

Stealthy patterns can be tuned by varying the parameter \( \chi \): ratio of number of constrained degrees of freedom to the total number of degrees of freedom, \( d(N - 1) \).
Creation of Disordered Stealthy Ground States
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One class of stealthy potentials involves the following power-law form:

\[ \tilde{v}(k) = v_0 \left(1 - \frac{k}{K}\right)^m \Theta(K - k), \]

where \( n \) is any whole number. The special case \( n = 0 \) is just the simple step function.

In the large-system (thermodynamic) limit with \( m = 0 \) and \( m = 4 \), we have the following large-\( r \) asymptotic behavior, respectively:

\[ v(r) \sim \frac{\cos(r)}{r^2} \quad (m = 0) \]
\[ v(r) \sim \frac{1}{r^4} \quad (m = 4) \]

While the specific forms of these stealthy potentials lead to the same convergent ground-state energies, this will not be the case for the pressure and other thermodynamic quantities.
From various initial distributions of $N$ points, found the energy minimizing configurations (with extremely high precision) using optimization techniques.
Creation of Disordered Stealthy Ground States

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- For $0 \leq \chi < 0.5$, stealthy ground states are highly degenerate, disordered and isotropic. Success rate to achieve disordered ground states is 100%.

As $\chi$ increases, short-range order increases. This suggests new order metric:

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\tau = \frac{1}{(2\pi)^d D^d} \int_{|k| \leq K} [S(k) - 1]^2 dk,
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Animations
Creation of Disordered Stealthy Ground States

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  ![Images showing configurations](a) $\chi = 0.04167$  
  ![Images showing configurations](b) $\chi = 0.41071$

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  Animations

- Stacked-slider phases: aperiodic, anisotropic metastable states with long-range orientational order for $\chi > 0.5$ and $d \geq 2$ (Zhang, Stillinger & Torquato, PRE 2015).
Stealthy Stacked-Slider Phases

Figure 1: Three examples of stealthy stacked-slider phases in $\mathbb{R}^2$ that are part of the ground-state manifold. Left panel: Horizontal rows in the square lattice are coherently translated with respect to one another. Middle panel: Horizontal rows in the square lattice are randomly translated with respect to one another. Right panel: Horizontal stealthy disordered stackings of disordered stealthy 1D configurations.

Zhang, Stillinger & Torquato, PRE (2015)
Stealthy Disordered Ground States and Novel Materials

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- High-density transparent stealthy disordered materials Leseur et al. (2016).
Nontrivial: Dimensionality of the configuration space depends on the number density $\rho$ (or $\chi$) and there is a multitude of ways of sampling the ground-state manifold, each with its own probability measure. Which ensemble? How are entropically favored states determined?

Derived general exact relations for thermodynamic properties that apply to any ground-state ensemble as a function of $\rho$ in any $d$ and showed how disordered degenerate ground states arise.
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From previous considerations, we that an important contribution to \( S(k) \) is a simple hard-core step function \( \Theta(k - K) \), which can be viewed as a disordered hard-sphere system in Fourier space in the limit that \( \chi \sim 1/\rho \) tends to zero, i.e., as the number density \( \rho \) tends to infinity.

That the structure factor must have the behavior

\[
S(k) \rightarrow \Theta(k - K), \quad \chi \rightarrow 0
\]

is perfectly reasonable; it is a perturbation about the ideal-gas limit in which \( S(k) = 1 \) for all \( k \).
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We make the ansatz that for sufficiently small \( \chi \), \( S(k) \) in the canonical ensemble for a stealthy potential can be mapped to \( g_2(r) \) for an effective disordered hard-sphere system for sufficiently small density.
Pseudo-Hard Spheres in Fourier Space

Let us define

$$\tilde{H}(k) \equiv \rho \tilde{h}(k) = h_{HS}(r = k)$$

There is an Ornstein-Zernike integral eq. that defines FT of appropriate direct correlation function, $\tilde{C}(k)$:

$$\tilde{H}(k) = \tilde{C}(k) + \eta \tilde{H}(k) \otimes \tilde{C}(k),$$

where $\eta$ is an effective packing fraction. Therefore,

$$H(r) = \frac{C(r)}{1 - (2\pi)^d \eta C(r)}.$$

This mapping enables us to exploit the well-developed accurate theories of standard Gibbsian disordered hard spheres in direct space.
Remark About Excited States

For a system in equilibrium, the compressibility relation relates the isothermal compressibility $\kappa_T$ to the structure factor at $k = 0$:

$$ S(0) = \rho k_B T \kappa_T, $$

where $k_B$ is Boltzmann’s constant. Because $\kappa_T$ is is bounded according to $\kappa_T = \rho^{-2}$ for finite $\rho$, $S(0) = 0$ because $T = 0$, which clearly must be the case for stealthy ground states.

Now consider excited states infinitesimally close to the stealthy ground states, i.e., when temperature $T$ is positive and infinitesimally small.

Under the assumption that the structure of such excited states will be infinitesimally near the ground-state configurations, we can estimate to an excellent approximation how $S(0)$ varies with $T$ for such excited states:

$$ S(0) \sim c(d) \chi T $$

in units $k_B = v_0 = 1$. Moreover, it is expected that this positive value of $S(0)$ will be the uniform value of $S(k)$ for $0 \leq k \leq K$.

This behavior of $S(k)$ has indeed been verified by molecular dynamics simulations in the canonical ensemble:
General Hyperuniform Scaling Behaviors

- Fourier representation of number variance:
  \[
  \sigma^2(R) = \langle N(R) \rangle \left[ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} S(k) \tilde{\alpha}(k; R) dk \right]
  \]

- Consider hyperuniform systems characterized by a structure factor with a radial power-law of the form

  \[ S(k) \sim |k|^{\alpha}, \quad \alpha > 0. \]

Limits \( \alpha \to 0 \) and \( \alpha \to \infty \) correspond to Poisson and crystal (or stealthy) systems.

- It follows that the number variance \( \sigma^2(R) \) increases for large \( R \) asymptotically as \( \text{(Zachary and Torquato, 2011)} \)

  \[
  \sigma^2(R) \sim \begin{cases} 
  R^{d-1} \ln R, & \alpha = 1 \\
  R^{d-\alpha}, & \alpha < 1 \\
  R^{d-1}, & \alpha > 1 
  \end{cases} \quad (R \to \infty).
  \]

- Multitude of scaling laws!

- Until recently, all known hyperuniform systems pertained to \( \alpha \geq 1 \).
Targeted Spectra $S \sim k^\alpha$

Configurations are ground states of an interacting many-particle system with up to two-, three- and four-body interactions.
Targeted Spectra $S \sim k^\alpha$ with $\alpha \geq 1$

Uche, Stillinger & Torquato (2006)

Figure 2: One of them is for $S(k) \sim k^6$ and other for $S(k) \sim k$. 
**Targeted Spectra** $S \sim k^{\alpha}$ with $\alpha < 1$

Zachary & Torquato (2011)

![Diagram](image)

**Figure 3:** Both configurations exhibit strong local clustering of points and possess a highly irregular local structure; however, only one of them is hyperuniform (with $S \sim k^{1/2}$).

**“Perfect Glasses”**

- By targeting a certain class of $S(k)$, interactions are devised that eliminate the possibilities of crystalline and quasicrystalline phases, while creating mechanically stable, hyperuniform amorphous glasses down to absolute zero.

Amorphous Silicon is Nearly Hyperuniform

Highly sensitive transmission X-ray scattering measurements performed at Argonne on amorphous-silicon (a-Si) samples reveals that they are nearly hyperuniform with $S(0) = 0.0075$.

Long, Roorda, Hejna, Torquato, and Steinhardt (2013)

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Increasing the degree of hyperuniformity of a-Si appears to be correlated with a larger electronic band gap (Hejna, Steinhardt and Torquato, 2013).
Structural Glasses and Growing Length Scales

Important question in glass physics: Do growing relaxation times under supercooling have accompanying growing structural length scales? Lubchenko & Wolynes (2006); Berthier et al. (2007); Karmakar, Dasgupta & Sastry (2009); Chandler & Garrahan (2010); Hocky, Markland & Reichman (2012)
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We studied glass-forming liquid models that support an alternative view: existence of growing static length scales (due to increase of the degree of hyperuniformity) as the temperature $T$ of the supercooled liquid is decreased to and below $T_g$ that is intrinsically nonequilibrium in nature.
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- We studied glass-forming liquid models that support an alternative view: existence of **growing static length scales** (due to increase of the **degree of hyperuniformity**) as the temperature $T$ of the supercooled liquid is decreased to and below $T_g$ that is intrinsically **nonequilibrium** in nature.

The degree of deviation from thermal equilibrium is determined from a **nonequilibrium index**

$$X = \frac{S(k = 0)}{\rho k_B T \kappa_T} - 1,$$

which increases upon supercooling. Marcotte, Stillinger & Torquato (2013)
Hyperuniformity of Disordered Two-Phase Materials

Hyperuniformity concept was generalized to the case of heterogeneous materials: phase volume fraction fluctuates within a spherical window of radius $R$ (Zachary and Torquato, 2009).
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For typical disordered media, volume-fraction variance $\sigma^2_V(R)$ for large $R$ goes to zero like $R^{-d}$.

For hyperuniform disordered two-phase media, $\sigma^2_V(R)$ goes to zero faster than $R^{-d}$, equivalent to following condition on spectral density $\tilde{\chi}_V(k)$:

$$\lim_{|k| \to 0} \tilde{\chi}_V(k) = 0.$$
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Interfacial-area fluctuations play an important role in static and surface-area evolving structures. Here we define $\sigma^2_S(R)$ and hyperuniformity condition is (Torquato, PRE, 2016)

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Disordered hyperuniform two-phase systems can be designed with targeted spectral functions (Torquato, J. Phys.: Cond. Mat., 2016).

For example, consider the following hyperuniform functional forms in 2D and 3D:
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For example, consider the following hyperuniform functional forms in 2D and 3D:

\[ \tilde{\chi}_V(k) \sim \begin{cases} d=3 & \text{for } k < 1 \\ d=2 & \text{for } k > 1 \end{cases} \]

The following is a 2D realization:
Other Generalizations of Hyperuniformity

Recently considered (Torquato, PRE 2016)

- **Random scalar fields**: Concentration and temperature fields in random media and turbulent flows, laser speckle patterns, and temperature fluctuations associated with CMB.

- **Random vector fields**: Random media (e.g., heat, current, electric, magnetic and velocity vector fields) and turbulence.

- **Structurally anisotropic materials**: Many-particle systems and random media that are statistically anisotropic.
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- Directional hyperuniformity: For unit vector $k_Q$ and scalar $t$,
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  \lim_{t \to 0} \tilde{\Psi}_{ij}(t k_Q) = 0
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Is there a many-particle system with following anisotropic scattering pattern?

![Image of S(k) graph](image-url)
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Hyperuniformity in compact spaces: Grabner et al. (2016)
CONCLUSIONS

- Hyperuniformity provides a unified means of categorizing and characterizing crystals, quasicrystals and special correlated disordered systems.
- The degree of hyperuniformity provides an order metric for the extent to which large-scale density fluctuations are suppressed in such systems.
- Disordered hyperuniform materials are new ideal states of disordered matter that can be endowed with novel physical properties that we are only beginning to discover.
- Hyperuniformity has connections to physics and materials science (e.g., ground states, quantum systems, random matrices, novel materials, etc.), mathematics (e.g., geometry and number theory), and biology.
- Hyperuniform scalar and vector fields as well as directional hyperuniform materials represent exciting new extensions.
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Collaborators

- Paul Chaikin (NYU)
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