Adjoint $\text{QCD}_2$ at Finite $N$

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(Joint work with Igor Klebanov, Loki Lin & Silviu Pufu)

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**Adjoint QCD**

- SU($N \to \infty$) Yang-Mills in 1+1D confines quarks [’t Hooft 1974]

- Adding an adjoint Majorana fermion:
  
  $$ S = \int d^2x \left( -\frac{1}{4} g^2 \text{Tr} \left( F_{\mu\nu} F_{\mu\nu} \right) + i \frac{1}{2} \bar{\Psi} \gamma^\mu /D^\mu \Psi - \frac{1}{2} m_{\text{adj}} \bar{\Psi} \Psi \right) $$

- Gives a dynamical particle transforming in the adjoint

- Screens quarks when the adjoint is massless [Gross et al. 1996; Komargodski et al. 2021; Dempsey et al. 2021]

- What happens at finite $N$?

- DLCQ works in principle [Antonuccio and Pinsky 1998], but difficult to find an orthonormal basis for small $N$

- Here: a new approach enabling high-resolution spectra at small $N$
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S = \int d^2x \left( -\frac{1}{4g^2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + \frac{i}{2} \bar{\psi} \hat{D} \psi - \frac{1}{2} m_{\text{adj}} \bar{\psi} \psi \right)
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![Potential](image)
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<td>At leading order, already orthogonal &amp; easy to normalize</td>
<td>Not orthogonal, very difficult to calculate inner products</td>
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Null States

- One approach: ignore inner products and use non-orthonormal basis
- Works fine for small $K$ at large but finite $N$ [Antonuccio and Pinsky 1998]

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\text{Problem: at small } N, \ SU(N) \text{ trace relations imply many null states.}
\]

For instance, in $SU(2)$,

\[
\text{Tr} (B^\dagger(1) B^\dagger(1) B^\dagger(3) B^\dagger(3)) = (\text{Tr} (B^\dagger(1) B^\dagger(3)))^2
\]

We could prove this using inner products, but these are hard to compute.

Representation theory counts the number of physical states:

- $SU(2)$:
  - $K = 20, 21, 22, 23, 24, 25$
- $SU(3)$:
  - $K = 31, 35, 40, 51, 63, 70$
- $SU(4)$:
  - $K = 380, 502, 658, 888, 1,188, 1,544$

Large $N$

- $2,530, 4,057, 6,525, 10,630, 17,262, 27,799$

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<td>SU(4)</td>
<td>1,328</td>
<td>1,927</td>
<td>2,794</td>
<td>4,100</td>
<td>5,947</td>
<td>8,476</td>
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\[ \{X_1, X_2\} = \text{Tr} (X_1 X_2) \mathbb{1} \]  (*)
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\[ (*) \]

Taking \( X_1 = B^\dagger(1) \) and \( X_2 = [B^\dagger(1), B^\dagger(3)] - \text{Tr} \left( B^\dagger(1) B^\dagger(3) \right) \mathbb{1} \), we find

\[ B^\dagger(1)^2 B^\dagger(3) - B^\dagger(3) B^\dagger(1)^2 = 2B^\dagger(1) \text{Tr} \left( B^\dagger(1) B^\dagger(3) \right) \]

Contracting with \( B^\dagger(3) \), we then have a proof of

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that did not require evaluating any inner products.
Cayley-Hamilton Relations

- In SU(2), one can show

\[ \{X_1, X_2\} = \Tr(X_1 X_2) \mathbb{1} \]  

\[ (\ast) \]

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- Contracting with \( B^\dagger(3) \), we then have a proof of

\[ \Tr(B^\dagger(1) B^\dagger(1) B^\dagger(3) B^\dagger(3)) = (\Tr(B^\dagger(1) B^\dagger(3)))^2 \]

that did not require evaluating any inner products

- The relation \((\ast)\) is generalized by the Cayley-Hamilton theorem applied to generators of any SU\((N)\)

- All null relations follow from these Cayley-Hamilton relations
DLCQ algorithm for small $N$

1. For each $K$, enumerate ways to split into momentum modes (e.g. $K = 15$ into 5 $B^\dagger(1)$’s and 2 $B^\dagger(5)$’s)
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6. Diagonalize to find spectra at each $K$
Spectrum for SU(3)
Spectrum for $SU(4)$
Spectrum for Large $N$

\[ \frac{\pi M^2}{g^2 N} \]

- Bosons
- Fermions

Values:
- 5.7
- 17.2
- 22.9
- 10.8
- 30
- 35
- 40
- 45
- 50

Parameters:
- $1/K$
$\frac{\pi \Delta M^2}{g^2 N}$

- $M^2 = 17.2$ Fermion
- $M^2 = 10.8$ Boson
- $M^2 = 5.7$ Fermion
SU(2): Specialized Algorithm

- For SU(2), almost all large $N$ states are null. At $K = 50$, there are 2,778 SU(2) states out of 4,666,298,795 large $N$ states.
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We can do much better by treating the SU(2) adjoint as an SO(3) fundamental and enumerating states in terms of SO(3) invariant tensors.
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Calculating $P^-$ gets more difficult, but still far easier than the Cayley-Hamilton method would be at large $K$. 
Results: $SU(2)$
Adjoint QCD$_2$ can be studied at small $N$ via DLCQ using Cayley-Hamilton relations.

- The $\frac{1}{N^2}$ corrections to bound states are small, and can be trusted down to $N = 3$.
- The SU(2) theory is distinct, because we can think of it as SO(3) with a fundamental.

Further questions:
- Can we better understand the small $\frac{1}{N^2}$ corrections?
- What happens when we add fundamental fermions?