Universal Randomness in 2D

Jason Miller and Scott Sheffield

October 3, 2018
OVERVIEW

We explore probability measures on spaces of *continuum objects* (paths, trees, surfaces, etc.) that are *canonical* (in sense that they are uniquely characterized by symmetries) and *universal* (in sense that they arise as limits of many different discrete random objects).
We explore probability measures on spaces of *continuum objects* (paths, trees, surfaces, etc.) that are *canonical* (in sense that they are uniquely characterized by symmetries) and *universal* (in sense that they arise as limits of many different discrete random objects).

Objects like this have a long history within both physics and mathematics.
We explore probability measures on spaces of continuum objects (paths, trees, surfaces, etc.) that are canonical (in sense that they are uniquely characterized by symmetries) and universal (in sense that they arise as limits of many different discrete random objects).

Objects like this have a long history within both physics and mathematics.

The discrete objects tend to be easier to understand and simulate.
OVERVIEW

- We explore probability measures on spaces of *continuum objects* (paths, trees, surfaces, etc.) that are *canonical* (in sense that they are uniquely characterized by symmetries) and *universal* (in sense that they arise as limits of many different discrete random objects).

- Objects like this have a long history within both physics and mathematics.

- The discrete objects tend to be easier to understand and simulate.

- But the continuum objects have additional symmetries.
We explore probability measures on spaces of continuum objects (paths, trees, surfaces, etc.) that are canonical (in sense that they are uniquely characterized by symmetries) and universal (in sense that they arise as limits of many different discrete random objects).

Objects like this have a long history within both physics and mathematics.

The discrete objects tend to be easier to understand and simulate.

But the continuum objects have additional symmetries.

Proving that the discrete analogs approximate the continuum analogs is a perpetual challenge. (Dozens of known and dozens of unknown results...)

This talk gives a general overview of the subject.
We explore probability measures on spaces of \textit{continuum objects} (paths, trees, surfaces, etc.) that are \textit{canonical} (in sense that they are uniquely characterized by symmetries) and \textit{universal} (in sense that they arise as limits of many different discrete random objects).

Objects like this have a long history within both physics and mathematics.

The discrete objects tend to be easier to understand and simulate.

But the continuum objects have additional symmetries.

Proving that the discrete analogs approximate the continuum analogs is a perpetual challenge. (Dozens of known and dozens of unknown results...)

This talk gives a general overview of the subject.
Some references

1. Exploration trees and conformal loop ensembles (S. 2006)
2. Contour lines of the two-dimensional discrete GFF (Schramm, S. 2006)
3. Liouville quantum gravity and KPZ (Duplantier, S. 2008)
4. A contour line of the continuum Gaussian free field (Schramm, S. 2008)
5. Conformal Loop Ensembles: The Markovian characterization and the loop-soup construction (S., Werner, 2010)
7. Quantum gravity and inventory accumulation (S., 2011)
8. Imaginary Geometry I-IV (Miller, S., 2012-2013)
9. Quantum Loewner Evolution (Miller, S. 2013)
10. Liouville quantum gravity as a mating of trees (Duplantier, Miller, S. 2014)
11. Liouville quantum gravity spheres as matings of finite trees (Miller, S 2015)
12. An axiomatic characterization of the Brownian map (Miller, S. 2015)
13. Liouville quantum gravity and the Brownian map I-III (Miller, S. 2015-2016)
Random trees
Random processes
Random surfaces
Random non-self-crossing paths
Random growth
Random (generalized) functions
Random loop ensembles

Continuum objects and relationships (all have discrete analogs)
Overview

Part I: Cast of Characters: *What are the most fundamental 2D random objects?*

1. **Universal random trees**: Brownian motion, continuum random tree
2. **Universal random surfaces**: quantum gravity, planar maps, string theory, CFT
3. **Universal random paths**: walks, interfaces, Schramm-Loewner evolution, CFT
4. **Universal random growth**: Eden model, DLA, DBM

Part II: Drama: *How are the characters related to each other?*

1. **Welding random surfaces**: a calculus of random surfaces and SLE seams
2. **Mating random trees**: tree plus tree (conformally mated) equals surface plus path
3. **Random growth on random surfaces**: dendrites, dragons, surprising tractability
4. **Mating random trees produced by a snake**: metric spaces and the Brownian map
5. **Two “universal random surfaces” are the same**: Brownian map equals Liouville quantum gravity with parameter $\gamma = \sqrt{8/3}$ (a.k.a. “pure quantum gravity”).
PROLOGUE:
NON-RANDOM FRACTALS
FROM COMPLEX DYNAMICS
Google search for Julia sets
Julia sets (Julia, 1918), popularized in 1980’s.
FRACTALS FROM COMPLEX DYNAMICS

- Julia sets (Julia, 1918), popularized in 1980’s
- Consider map \( \phi(z) = z^2 \).

Published 1989, by Roger T. Stevens
Julia sets (Julia, 1918), popularized in 1980’s
Consider map $\phi(z) = z^2$.
Maps $\mathbb{C} \setminus \overline{D}$ conformally to self (2 to 1) where $D$ is unit disc. Repeated iteration takes points in $\mathbb{C} \setminus \overline{D}$ to $\infty$, leaves others bounded.

K is another compact set with connected hull, can construct a similar (2 to 1) conformal map $\phi_K$ from $\mathbb{C} \setminus K$ to itself.
Might expect more intricate sets $K$ to yield more intricate maps. But suppose we take $\phi_K(z) = z^2 + c$ and let $K$ be set of points remaining bounded under repeated iteration.

$K$ is a (filled) Julia set. Can “mate” Julia sets to form sphere (Douady 1983, Milnor 1994, see Arnaud Ch´eritat’s animations).
Popular lexicon: chaos theory, butterfly effect, fractal, self-similar. What about random fractals, only self similar in law?
Julia sets (Julia, 1918), popularized in 1980’s

Consider map $\phi(z) = z^2$.

Maps $\mathbb{C} \setminus \overline{D}$ conformally to self (2 to 1) where $D$ is unit disc. Repeated iteration takes points in $\mathbb{C} \setminus \overline{D}$ to $\infty$, leaves others bounded.

If $K$ is another compact set with connected hull, can construct a similar (2 to 1) conformal map $\phi_K$ from $\mathbb{C} \setminus \overline{K}$ to itself.
Julia sets (Julia, 1918), popularized in 1980’s

Consider map $\phi(z) = z^2$.

Maps $\mathbb{C} \setminus \overline{D}$ conformally to self (2 to 1) where $D$ is unit disc. Repeated iteration takes points in $\mathbb{C} \setminus \overline{D}$ to $\infty$, leaves others bounded.

If $K$ is another compact set with connected hull, can construct a similar (2 to 1) conformal map $\phi_K$ from $\mathbb{C} \setminus \overline{K}$ to itself.

Might expect more intricate sets $K$ to yield more intricate maps. But suppose we take $\phi_K(z) = z^2 + c$ and let $K$ be set of points remaining bounded under repeated iteration.

Published 1989, by Roger T. Stevens
Julia sets (Julia, 1918), popularized in 1980’s

Consider map \( \phi(z) = z^2 \).

Maps \( \mathbb{C} \setminus \overline{D} \) conformally to self (2 to 1) where \( D \) is unit disc. Repeated iteration takes points in \( \mathbb{C} \setminus \overline{D} \) to \( \infty \), leaves others bounded.

If \( K \) is another compact set with connected hull, can construct a similar (2 to 1) conformal map \( \phi_K \) from \( \mathbb{C} \setminus \overline{K} \) to itself.

Might expect more intricate sets \( K \) to yield more intricate maps. But suppose we take \( \phi_K(z) = z^2 + c \) and let \( K \) be set of points remaining bounded under repeated iteration.

\( K \) is a (filled) Julia set. Can “mate” Julia sets to form sphere (Douady 1983, Milnor 1994, see Arnaud Chéritat’s animations).
Julia sets (Julia, 1918), popularized in 1980’s

Consider map \( \phi(z) = z^2 \).

Maps \( \mathbb{C} \setminus \bar{D} \) conformally to self (2 to 1) where \( D \) is unit disc. Repeated iteration takes points in \( \mathbb{C} \setminus \bar{D} \) to \( \infty \), leaves others bounded.

If \( K \) is another compact set with connected hull, can construct a similar (2 to 1) conformal map \( \phi_K \) from \( \mathbb{C} \setminus \bar{K} \) to itself.

Might expect more intricate sets \( K \) to yield more intricate maps. But suppose we take \( \phi_K(z) = z^2 + c \) and let \( K \) be set of points remaining bounded under repeated iteration.

\( K \) is a (filled) **Julia set**. Can “mate” Julia sets to form sphere (Douady 1983, Milnor 1994, see Arnaud Chéritat’s animations).

**Popular lexicon:** chaos theory, butterfly effect, fractal, self-similar. What about *random* fractals, only self similar in law?
Part I: CAST OF CHARACTERS

A Trees
B Simple curves, non-simple curves, space-filling curves
C Surfaces
D Growth
This is the easiest “universal” random fractal to explain.
This is the easiest "universal" random fractal to explain.

Aldous (1993) constructs continuum random tree (CRT) from a Brownian excursion. To produce tree, start with graph of Brownian excursion and then identify points connected by horizontal line segment that lies below graph except at endpoints. Result is a random matric space.
This is the easiest “universal” random fractal to explain.

Aldous (1993) constructs **continuum random tree** (CRT) from a Brownian excursion. To produce tree, start with graph of Brownian excursion and then identify points connected by horizontal line segment that lies below graph except at endpoints. Result is a random matric space.

Discrete analog: Consider a tree embedded in the plane with \( n \) edges and a distinguished root. As one traces the outer boundary of the tree clockwise, distance from root performs a simple walk on \( \mathbb{Z}_+ \) with \( 2n \) steps, starting and ending at 0.
This is the easiest “universal” random fractal to explain.

Aldous (1993) constructs **continuum random tree** (CRT) from a Brownian excursion. To produce tree, start with graph of Brownian excursion and then identify points connected by horizontal line segment that lies below graph except at endpoints. Result is a random metric space.

Discrete analog: Consider a tree embedded in the plane with $n$ edges and a distinguished root. As one traces the outer boundary of the tree clockwise, distance from root performs a simple walk on $\mathbb{Z}_+$ with $2n$ steps, starting and ending at 0.

Simple bijection between rooted planar trees and walks of this type.
This is the easiest “universal” random fractal to explain.

Aldous (1993) constructs **continuum random tree** (CRT) from a Brownian excursion. To produce tree, start with graph of Brownian excursion and then identify points connected by horizontal line segment that lies below graph except at endpoints. Result is a random matric space.

Discrete analog: Consider a tree embedded in the plane with \( n \) edges and a distinguished root. As one traces the outer boundary of the tree clockwise, distance from root performs a simple walk on \( \mathbb{Z}_+ \) with \( 2n \) steps, starting and ending at 0.

Simple bijection between rooted planar trees and walks of this type.

CRT is in some sense the “uniformly random planar tree” of a given size.
RANDOM PATHS

Given a simply connected planar domain $D$ with boundary points $a$ and $b$ and a parameter $\kappa \in [0, \infty)$, the Schramm-Loewner evolution $\text{SLE}_\kappa$ is a random non-self-crossing path in $\overline{D}$ from $a$ to $b$.

The parameter $\kappa$ roughly indicates how “windy” the path is. Would like to argue that SLE is in some sense the “canonical” random non-self-crossing path. What symmetries characterize SLE?
If \( \phi \) conformally maps \( D \) to \( \tilde{D} \) and \( \eta \) is an \( \text{SLE}_\kappa \) from \( a \) to \( b \) in \( D \), then \( \phi \circ \eta \) is an \( \text{SLE}_\kappa \) from \( \phi(a) \) to \( \phi(b) \) in \( \tilde{D} \).
Markov Property

Given $\eta$ up to a stopping time $t$...

law of remainder is SLE in $D \setminus \eta[0, t]$ from $\eta(t)$ to $b$. 
THEOREM [Oded Schramm]: Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$. 

Explicit construction:

An SLE path $\gamma$ from $0$ to $\infty$ in the complex upper half plane $\mathbb{H}$ can be defined in an interesting way: given path $\gamma$ one can construct conformal maps $g_t: \mathbb{H} \setminus \gamma(\{0, t\}) \to \mathbb{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z \to \infty} g_t(z) - z = 0$). In SLE $\kappa$, one defines $g_t$ via an ODE (which makes sense for each fixed $z$):

$$\partial_t g_t(z) = 2g_t(z) - W_t,$$

$g_0(z) = z$,

where $W_t = \sqrt{\kappa}B_t = \text{LAW} B_\kappa t$ and $B_t$ is ordinary Brownian motion.
THEOREM [Oded Schramm]: Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.

Explicit construction: An SLE path $\gamma$ from 0 to $\infty$ in the complex upper half plane $H$ can be defined in an interesting way: given path $\gamma$ one can construct conformal maps $g_t : H \setminus \gamma([0, t]) \to H$ (normalized to look like identity near infinity, i.e., $\lim_{z \to \infty} g_t(z) - z = 0$). In $\text{SLE}_\kappa$, one defines $g_t$ via an ODE (which makes sense for each fixed $z$):

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $W_t = \sqrt{\kappa} B_t =_{\text{law}} B_{\kappa t}$ and $B_t$ is ordinary Brownian motion.
SLE phases [Rohde, Schramm]

\[ \kappa \leq 4 \]

\[ \kappa \in (4, 8) \]

\[ \kappa \geq 8 \]
Radial Schramm-Loewner evolution (SLE)

- In *radial SLE* path grows from boundary of domain to center.
Radial Schramm-Loewner evolution (SLE)

- In radial SLE path grows from boundary of domain to center.
- Modified version allow growth from multiple boundary points (or a continuum of points) at once.

\[
\frac{\partial}{\partial t} g_t(z) = g_t(z) \xi_t + g_t(z) \overline{\xi_t} - g_t(z) \overline{g_t(z)}
\]

where \( \xi_t = e^{i \sqrt{\kappa} B_t} \).

Radial measure-driven Loewner evolution:

\[
\frac{\partial}{\partial t} g_t(z) = \int g_t(z) x + g_t(z) \overline{x} - g_t(z) \overline{g_t(x)} dm_t(x)
\]

where, for each \( g_t \), \( m_t \) is a measure on the complex unit circle.
Radial Schramm-Loewner evolution (SLE)

- In *radial SLE* path grows from boundary of domain to center.
- Modified version allow growth from multiple boundary points (or a continuum of points) at once.
- This will be important when we think about continuum growth models.
Radial Schramm-Loewner evolution (SLE)

- In radial SLE path grows from boundary of domain to center.
- Modified version allow growth from multiple boundary points (or a continuum of points) at once.
- This will be important when we think about continuum growth models.
- **Radial SLE:** \( \partial_t g_t(z) = g_t(z) \frac{\xi_t + g_t(z)}{\xi_t - g_t(z)} \) where \( \xi_t = e^{i\sqrt{\kappa}B_t} \).
Radial Schramm-Loewner evolution (SLE)

- In *radial SLE* path grows from boundary of domain to center.
- Modified version allow growth from multiple boundary points (or a continuum of points) at once.
- This will be important when we think about continuum growth models.
- **Radial SLE:** \( \partial_t g_t(z) = g_t(z) \frac{\xi_t + g_t(z)}{\xi_t - g_t(z)} \) where \( \xi_t = e^{i \sqrt{\kappa} B_t} \).
- **Radial measure-driven Loewner evolution:** \( \partial_t g_t(z) = \int g_t(z) \frac{x + g_t(z)}{x - g_t(z)} \, dm_t(x) \) where, for each \( g \), \( m_t \) is a measure on the complex unit circle.
Continuum space-filling path
Start out with a sheet of paper
Get out pen and ruler
Measure and mark squares of equal size
Get out scissors
Cut into squares
RANDOM SURFACES

Get out bottle of glue
Attach squares along boundaries with glue to form a surface “without holes.”
What is the structure of a typical quadrangulation when the number of faces is large?
What is the structure of a typical quadrangulation when the number of faces is large?
Random quadrangulation with 25,000 faces

(Simulation due to J.F. Marckert)
1. First studied by Tutte in 1960s while working on the four color theorem.

(Simulation due to J.F. Marckert)
Background

1. First studied by Tutte in 1960s while working on the four color theorem.

2. Many variants (triangulations, quadrangulations, etc.) Some come equipped with extra statistical physics structure (a distinguished spanning tree, a general distinguished edge subset, a “spin” function on vertices, etc.)

(Simulation due to J.F. Marckert)
1. First studied by Tutte in 1960s while working on the four color theorem.

2. Many variants (triangulations, quadrangulations, etc.) Some come equipped with extra statistical physics structure (a distinguished spanning tree, a general distinguished edge subset, a “spin” function on vertices, etc.)

3. Can be interpreted as Riemannian manifolds with conical singularities.

(Simulation due to J.F. Marckert)
Background

1. First studied by Tutte in 1960s while working on the four color theorem.

2. Many variants (triangulations, quadrangulations, etc.) Some come equipped with extra statistical physics structure (a distinguished spanning tree, a general distinguished edge subset, a “spin” function on vertices, etc.)

3. Can be interpreted as Riemannian manifolds with conical singularities.

4. Converges in law in Gromov-Hausdorff sense to random metric space called Brownian map, homeomorphic to the 2-sphere, Hausdorff dimension 4 (established in several works by subsets of Chaissang, Schaefer, Le Gall, Paulin, Miermont)

(Simulation due to J.F. Marckert)
Background

1. First studied by Tutte in 1960s while working on the four color theorem.

2. Many variants (triangulations, quadrangulations, etc.) Some come equipped with extra statistical physics structure (a distinguished spanning tree, a general distinguished edge subset, a “spin” function on vertices, etc.)

3. Can be interpreted as Riemannian manifolds with conical singularities.

4. Converges in law in Gromov-Hausdorff sense to random metric space called Brownian map, homeomorphic to the 2-sphere, Hausdorff dimension 4 (established in several works by subsets of Chaissang, Schaefer, Le Gall, Paulin, Miermont)

5. Important tool: Bijections encoding surface via pair of trees.
Random quadrangulation

Packed with Stephenson’s CirclePack.

Universal structure
Red tree
Red and blue trees
Red and blue trees alone do not determine the map structure
Random quadrangulation with red and blue trees
Path snaking between the trees. Encodes the trees and how they are glued together.
How was the graph embedded into $\mathbb{R}^2$?
Can subdivide each quadrilateral to obtain a triangulation without multiple edges.
Circle pack the resulting triangulation.

Packed with Stephenson’s CirclePack.
Circle pack the resulting triangulation.

Packed with Stephenson’s CirclePack.
Circle pack the resulting triangulation.

Packed with Stephenson’s CirclePack.
What is the “limit” of this embedding? Circle packings are related to conformal maps.

Packed with Stephenson’s CirclePack.
Conformal maps (from David Gu’s web gallery)

Riemann Uniformization

All metric surfaces can be conformally mapped to three canonical spaces, the sphere, the plane and the hyperbolic plane.

Genus zero closed surface
Picking a surface at random in the continuum

**Uniformization theorem:** every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $S^2$ in $\mathbb{R}^3$
Picking a surface at random in the continuum

**Uniformization theorem:** every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $S^2$ in $\mathbb{R}^3$

**Isothermal coordinates:** Metric for the surface takes the form $e^{\rho(z)} \, dz$ for some smooth function $\rho$ where $dz$ is the Euclidean metric.
Picking a surface at random in the continuum

**Uniformization theorem:** every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $S^2$ in $\mathbb{R}^3$

**Isothermal coordinates:** Metric for the surface takes the form $e^{\rho(z)} \, dz$ for some smooth function $\rho$ where $dz$ is the Euclidean metric.

⇒ Can parameterize the space of surfaces with smooth functions.

- If $\rho = 0$, get the same surface
- If $\Delta \rho = 0$, i.e. if $\rho$ is harmonic, the surface described is flat
Picking a surface at random in the continuum

**Uniformization theorem:** every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $S^2$ in $\mathbb{R}^3$.

**Isothermal coordinates:** Metric for the surface takes the form $e^{\rho(z)}dz$ for some smooth function $\rho$ where $dz$ is the Euclidean metric.

⇒ Can parameterize the space of surfaces with smooth functions.

▶ If $\rho = 0$, get the same surface
▶ If $\Delta \rho = 0$, i.e. if $\rho$ is harmonic, the surface described is flat

**Question:** Which measure on $\rho$? If we want our surface to be a perturbation of a flat metric, natural to choose $\rho$ as the canonical perturbation of a harmonic function.
The Gaussian free field

- The discrete Gaussian free field (DGFF) is a Gaussian random surface model.
The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.
- Measure on functions $h: D \to \mathbb{R}$ for $D \subseteq \mathbb{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbb{R}^{|D|}$:

\[
\frac{1}{\mathcal{Z}} \exp \left( -\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)
\]
The Gaussian free field

- The **discrete Gaussian free field (DGFF)** is a **Gaussian random surface** model.

- Measure on functions \( h: D \to \mathbb{R} \) for \( D \subseteq \mathbb{Z}^2 \) and \( h|_{\partial D} = \psi \) with density respect to Lebesgue measure on \( \mathbb{R}^{|D|} \):

  \[
  \frac{1}{Z} \exp \left( -\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)
  \]

- Natural perturbation of a harmonic function
The Gaussian free field

- The discrete Gaussian free field (DGFF) is a Gaussian random surface model.

- Measure on functions $h : D \to \mathbb{R}$ for $D \subseteq \mathbb{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbb{R}^{|D|}$:

$$\frac{1}{Z} \exp \left( -\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$

- Natural perturbation of a harmonic function

- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the Dirichlet inner product

$$\langle f, g \rangle_{\nabla} = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) \, dx.$$
The Gaussian free field

- The discrete Gaussian free field (DGFF) is a Gaussian random surface model.

- Measure on functions $h: D \to \mathbb{R}$ for $D \subseteq \mathbb{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbb{R}^{|D|}$:
  \[
  \frac{1}{Z} \exp \left( -\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)
  \]

- Natural perturbation of a harmonic function

- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the Dirichlet inner product
  \[
  (f, g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx.
  \]

- Continuum GFF not a function — only a generalized function
Determinant of Laplacian

- Can plot one point above for each planar map on sphere.
Can plot one point above for each planar map on sphere.

On discrete graph, Laplacian determinant counts number of trees.
Can plot one point above for each planar map on sphere.

On discrete graph, Laplacian determinant counts number of trees.

(Laplacian determinant)$^{-1/2}$ counts “number” (i.e., partition function) of GFF instances on graph.
Can plot one point above for each planar map on sphere.

On discrete graph, Laplacian determinant counts number of trees.

\((\text{Laplacian determinant})^{-1/2}\) counts “number” (i.e., partition function) of GFF instances on graph.

If you choose a planar map together with embedding in \(c\) dimensional space (a “discretized worldsheet), the probability of a given map is proportional to \((\text{Laplacian determinant})^{-c/2}\).
Can plot one point above for each planar map on sphere.

On discrete graph, Laplacian determinant counts number of trees.

\((\text{Laplacian determinant})^{-1/2}\) counts “number” (i.e., partition function) of GFF instances on graph.

If you choose a planar map together with embedding in \(c\) dimensional space (a “discretized worldsheet), the probability of a given map is proportional to \((\text{Laplacian determinant})^{-c/2}\).

Other kinds of decorations believed to correspond to different \(c\) values in scaling limit.
Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} \, dz$
  where $h$ is a GFF and $\gamma \in [0, 2)$

\[ \gamma = 0.5 \]

(Number of subdivisions)
Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} \, dz$
  where $h$ is a GFF and $\gamma \in [0, 2)$


- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0, \sqrt{2})$. Kahane, 1985, $\gamma \in [0, 2)$.

$\gamma = 0.5$

(Number of subdivisions)
Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} \, dz$ where $h$ is a GFF and $\gamma \in [0, 2)$
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0, \sqrt{2})$. Kahane, 1985, $\gamma \in [0, 2)$.
- Does not make literal sense since $h$ takes values in the space of distributions.

$\gamma = 0.5$

(Number of subdivisions)
Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} \, dz$
  where $h$ is a GFF and $\gamma \in [0, 2)$
- Rigorous construction of measure: Høegh-Krohn, 1971, $\gamma \in [0, \sqrt{2})$. Kahane, 1985, $\gamma \in [0, 2)$.
- Does not make literal sense since $h$ takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.

\( \gamma = 0.5 \)

(Number of subdivisions)
Liouville quantum gravity

- Liouville quantum gravity: \( e^{\gamma h(z)} \, dz \) where \( h \) is a GFF and \( \gamma \in [0, 2) \)
- Rigorous construction of measure: Høegh-Krohn, 1971, \( \gamma \in [0, \sqrt{2}) \). Kahane, 1985, \( \gamma \in [0, 2) \).
- Does not make literal sense since \( h \) takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.
- Areas of regions and lengths of curves are well defined.

\( \gamma = 1.0 \)
Liouville quantum gravity

- Liouville quantum gravity: \( \int e^{\gamma h(z)} \, dz \) where \( h \) is a GFF and \( \gamma \in [0, 2) \)
- Rigorous construction of measure: Høegh-Krohn, 1971, \( \gamma \in [0, \sqrt{2}) \). Kahane, 1985, \( \gamma \in [0, 2) \).
- Does not make literal sense since \( h \) takes values in the space of distributions.
- Can make sense of random area measure using a regularization procedure.
- Areas of regions and lengths of curves are well defined.

\( \gamma = 1.5 \)
Liouville quantum gravity

- Liouville quantum gravity: \( e^{\gamma h(z)} \, dz \)
  where \( h \) is a GFF and \( \gamma \in [0, 2) \)


- Rigorous construction of measure: Høegh-Krohn, 1971, \( \gamma \in [0, \sqrt{2}) \).
  Kahane, 1985, \( \gamma \in [0, 2) \).

- Does not make literal sense since \( h \) takes values in the space of distributions.

- Can make sense of random area measure using a regularization procedure.

- Areas of regions and lengths of curves are well defined.

\( \gamma = 2.0 \)

(Number of subdivisions)
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)

Associate with a graph \((V, E)\) i.i.d. exp(1) edge weights

Consider case that graph is \(Z^2\).

Question: Large scale behavior of shape of ball wrt perturbed metric?

Cox and Durrett (1981) showed that the macroscopic shape is convex

Computer simulations show that it is not a Euclidean disk

\(Z^2\) is not isotropic enough

Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if \(Z^2\) is replaced by the Voronoi tessellation associated with a Poisson process.
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V, E)\) i.i.d. \(\exp(1)\) edge weights

![Graph diagram with edge weights](image)

- Consider case that graph is \(\mathbb{Z}^2\).
- Question: Large scale behavior of shape of ball wrt perturbed metric?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
- \(\mathbb{Z}^2\) is not isotropic enough
- Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if \(\mathbb{Z}^2\) is replaced by the Voronoi tesselation associated with a Poisson process.
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph $(V, E)$ i.i.d. exp(1) edge weights

Cox and Durrett (1981) showed that the macroscopic shape is convex

Computer simulations show that it is not a Euclidean disk

$\mathbb{Z}^2$ is not isotropic enough

Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if $\mathbb{Z}^2$ is replaced by the Voronoi tesselation associated with a Poisson process.
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V, E)\) i.i.d. \(\text{exp}(1)\) edge weights
- Consider case that graph is \(\mathbb{Z}^2\).

Question:
Large scale behavior of shape of ball wrt perturbed metric?

Cox and Durrett (1981) showed that the macroscopic shape is convex

Computer simulations show that it is not a Euclidean disk

\(\mathbb{Z}^2\) is not isotropic enough

Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if \(\mathbb{Z}^2\) is replaced by the Voronoi tesselation associated with a Poisson process.
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)

- Associate with a graph \((V, E)\) i.i.d. exp(1) edge weights

- Consider case that graph is \(\mathbb{Z}^2\).

- **Question:** Large scale behavior of shape of ball wrt perturbed metric?

Cox and Durrett (1981) showed that the macroscopic shape is convex.

Computer simulations show that it is not a Euclidean disk.

\(\mathbb{Z}^2\) is not isotropic enough.

Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if \(\mathbb{Z}^2\) is replaced by the Voronoi tessellation associated with a Poisson process.
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V,E)\) i.i.d. \(\exp(1)\) edge weights
- Consider case that graph is \(\mathbb{Z}^2\).
- **Question:** Large scale behavior of shape of ball wrt perturbed metric?

Cox and Durrett (1981) showed that the macroscopic shape is convex. Computer simulations show that it is not a Euclidean disk. \(\mathbb{Z}^2\) is not isotropic enough. Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if \(\mathbb{Z}^2\) is replaced by the Voronoi tesselation associated with a Poisson process.
**RANDOM GROWTH**

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V, E)\) i.i.d. \(\exp(1)\) edge weights
- Consider case that graph is \(\mathbb{Z}^2\).
- **Question:** Large scale behavior of shape of ball wrt perturbed metric?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V, E)\) i.i.d. \(\exp(1)\) edge weights
- Consider case that graph is \(\mathbb{Z}^2\).
- **Question:** Large scale behavior of shape of ball wrt perturbed metric?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V, E)\) i.i.d. exp(1) edge weights
- Consider case that graph is \(\mathbb{Z}^2\).
- **Question:** Large scale behavior of shape of ball wrt perturbed metric?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
- \(\mathbb{Z}^2\) is not isotropic enough
RANDOM GROWTH

- FPP/Eden model growth, introduced by Eden (1961) and Hammersley and Welsh (1965)
- Associate with a graph \((V,E)\) i.i.d. \(\exp(1)\) edge weights
- Consider case that graph is \(\mathbb{Z}^2\).
- **Question:** Large scale behavior of shape of ball wrt perturbed metric?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
- \(\mathbb{Z}^2\) is not isotropic enough
- Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if \(\mathbb{Z}^2\) is replaced by the Voronoi tessellation associated with a Poisson process
Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Markovian formulation

Eden exploration

Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Sample the cluster $C_{n+1}$ from $C_n$ by selecting an edge uniformly at random on $\partial C_n$, and then adding the vertex which is attached to it. **VARIANT:** Choose locations from harmonic measure (DLA) or harmonic measure to $\eta$ power ($\eta$-DBM).
Euclidean Diffusion Limited Aggregation (DLA) introduced by Witten-Sander 1981.
DLA in nature: “A DLA cluster grown from a copper sulfate solution in an electrodeposition cell” (from Wikipedia)
DLA in nature: Magnese oxide patterns on the surface of a rock. (Halsey, Physics Today 2000)
DLA in nature: Magnese oxide patterns on the surface of a rock.
DLA in art: “High-voltage dielectric breakdown within a block of plexiglas” (from Wikipedia)
DLA in physics

Introduced by Witten and Sander in 1981 as a model for crystal growth. (Mineral deposits, Hele-Shaw flow, electrodeposition, lichen growth, lightning paths, coral, etc.)

An active area of research in physics for the last 33 years:

"diffusion limited aggregation"

About 11,700 results (0.05 sec)
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

- Does DLA have a “scaling limit”?
- Is the shape random at large scales?
- Does the macroscopic shape look like a tree?
- What is its asymptotic dimension? Simulation prediction: $\approx 1.71$ on $\mathbb{Z}^2$.
- Is there a universal isotropic continuum analog of DLA?

What about DLA on random planar maps and Liouville quantum gravity surfaces?
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

▶ Does DLA have a "scaling limit"?
▶ Is the shape random at large scales?
▶ Does the macroscopic shape look like a tree?
▶ What is its asymptotic dimension? Simulation prediction: \( \approx 1.71 \) on \( \mathbb{Z}^2 \)
▶ Is there a universal isotropic continuum analog of DLA?

What about DLA on random planar maps and Liouville quantum gravity surfaces?
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

▶ Does DLA have a “scaling limit”?
▶ Is the shape random at large scales?
▶ Does the macroscopic shape look like a tree?
▶ What is its asymptotic dimension? Simulation prediction: $\approx 1.71$ on $\mathbb{Z}^2$.
▶ Is there a universal isotropic continuum analog of DLA?

What about DLA on random planar maps and Liouville quantum gravity surfaces?
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

▶ Does DLA have a “scaling limit”?
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

▶ Does DLA have a “scaling limit”? 
▶ Is the shape random at large scales?
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

- Does DLA have a “scaling limit”? 
- Is the shape random at large scales? 
- Does the macroscopic shape look like a tree?
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

► Does DLA have a “scaling limit”? 
► Is the shape random at large scales?
► Does the macroscopic shape look like a tree?
► What is its asymptotic dimension? Simulation prediction: $\approx 1.71$ on $\mathbb{Z}^2$
DLA in math?

Not a lot of progress. (A related process called internal DLA is mathematically much more well understood.) Expected that (as with Eden model) lattice versions may have anisotropic features in limit.

Open questions

▶ Does DLA have a “scaling limit”?  
▶ Is the shape random at large scales?  
▶ Does the macroscopic shape look like a tree?  
▶ What is its asymptotic dimension? Simulation prediction: \( \approx 1.71 \) on \( \mathbb{Z}^2 \)  
▶ Is there a \textit{universal} isotropic continuum analog of DLA?

What about DLA on random planar maps and Liouville quantum gravity surfaces?
Part II: DRAMA
STORY A: IMAGINARY GEOMETRY: GENERALIZED FUNCTIONS & SPACE-FILLING CURVES
Rays of a Smooth Function

- $h$ smooth $[h(x, y) = x^2 + y^2]$
Rays of a Smooth Function

- $h$ smooth \[ h(x, y) = x^2 + y^2 \]
- Vector field $e^{ih(x,y)}$
Rays of a Smooth Function

- $h$ smooth \[ h(x, y) = x^2 + y^2 \]
- Vector field $e^{ih(x,y)}$
- A ray of $h$ is a flow line of $e^{ih}$, i.e. a solution to

\[
\frac{d}{dt} \eta(t) = e^{ih(\eta(t))}
\]
Rays of a Smooth Function

- $h$ smooth [$h(x, y) = x^2 + y^2$]
- Vector field $e^{ih(x,y)}$
- A $\theta$-angle ray of $h$ is a flow line of $e^{i(h+\theta)}$, i.e. a solution to

$$
\frac{d}{dt} \eta(t) = e^{i(h(\eta(t))+\theta)}
$$
Rays of a Smooth Function

- $h$ smooth $[h(x, y) = x^2 + y^2]$
- Vector field $e^{ih(x,y)}$
- A $\theta$-angle ray of $h$ is a flow line of $e^{i(h+\theta)}$, i.e. a solution to

$$\frac{d}{dt} \eta(t) = e^{i(h(\eta(t)) + \theta)}$$

- The rays of $h$ vary smoothly and monotonically with $\theta$ and are non-intersecting.
Rays of a Smooth Function

- $h$ smooth \([h(x, y) = x^2 + y^2]\)
- Vector field $e^{ih(x, y)}$
- A $\theta$-angle ray of $h$ is a flow line of $e^{i(h+\theta)}$, i.e. a solution to
  \[
  \frac{d}{dt} \eta(t) = e^{i(h(\eta(t)) + \theta)}
  \]
- The rays of $h$ vary smoothly and monotonically with $\theta$ and are non-intersecting.
- Facts:
  - Can make sense of flow lines when $h$ is GFF
  - These are forms of SLE
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 31.97$ [$\kappa = 1/256$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 11.23$ [$\kappa = 1/32$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 7.88$ [$\kappa = 1/16$]
Rays of $e^{ih/x}$, $h$ GFF, $\chi = 3.75$ [$\kappa = 1/4$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi = 1.5$ [$\kappa = 1$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 1.02$ [$\kappa = 3/2$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi = 0.71$ [$\kappa = 2$]
Rays of $e^{i\tilde{h}/\chi}$, $\tilde{h} = h + \beta \log |\cdot|$, $h$ GFF, $\beta < 0$
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 2.47$ \([\kappa = 1/2]\)
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]
Rays of $e^{ih/\chi}$, $h$ GFF, $\chi \approx 2.47$ [$\kappa = 1/2$]
Duality in the GFF: the SLE Light Cone

Flow lines with fixed angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. 
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; one direction change.
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; two direction changes.
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; three direction changes.
Duality in the GFF: the SLE Light Cone

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; four direction changes.
Duality in the GFF: the SLE Light Cone

**Theorem** (Miller, S.): The set of all points accessible by $\text{SLE}_\kappa$ flow lines ($\kappa \in (0,4)$) with angles restricted in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is an $\text{SLE}_{16/\kappa}$ process.
SLE_{64}(32; 32) Light Cone
STORY B:
SURFACE PLUS SURFACE = SURFACE PLUS CURVE

independence on both sides
WELDING RANDOM SURFACES

Can “weld” and “slice” special quantum surfaces called quantum wedges (with “weight” parameters indicating thickness) to obtain wedges (with other weights).

- Weight parameter $W = \gamma(\gamma + \frac{2}{\gamma} - \alpha)$ is additive under the welding operation.
- Interface between welding of independent wedges $\mathcal{W}_1, \mathcal{W}_2$ of weight $W_1$ and $W_2$ is an $\text{SLE}_\kappa(W_1 - 2; W_2 - 2)$ on combined surface.
- Glue canonical random surfaces, seam becomes canonical random path.
STORY C:

TREE PLUS TREE = SURFACE PLUS SPACE-FILLING CURVE

LHS independent or correlated, RHS independent
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)

Same for $C - Y$ yields an independent CRT

Glue the CRTs together by declaring points on the vertical lines to be equivalent

Q: What is the resulting structure?
A: Sphere with a space-filling path. A peanosphere.
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

$m_X \uparrow X_t$

Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)

$C - Y_t$

$X_t$

$\implies$ Sphere with a space-filling path.

$\implies$ Peanosphere.

$\implies$ Universal structure

October 3, 2018 48 / 68
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

- Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)
- Same for $C - Y_t$ yields an independent CRT

$\rightarrow$ Sphere with a space-filling path.

$\rightarrow$ Apéneanosphere.

Jason Miller and Scott Sheffield

Universal structure

October 3, 2018 48 / 68
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

- Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)
- Same for $C - Y_t$ yields an independent CRT
- Glue the CRTs together by declaring points on the vertical lines to be equivalent

Q:
What is the resulting structure?
A: Sphere with a space-filling path. A peanosphere.
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

- Identify points on the graph of $X$ if they are connected by a **horizontal** line which is below the graph; yields a continuum random tree (CRT)
- Same for $C - Y_t$ yields an independent CRT
- Glue the CRTs together by declaring points on the **vertical** lines to be equivalent

**Q:** What is the resulting structure?

**A:** Sphere with a space-filling path. A **Peanosphere**.
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

- Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)
- Same for $C - Y_t$ yields an independent CRT
- Glue the CRTs together by declaring points on the vertical lines to be equivalent

Q: What is the resulting structure?  
A: Sphere with a space-filling path.

$C - Y_t$

$X_t$

$t$
MATING RANDOM TREES

$X, Y$ independent Brownian excursions on $[0, 1]$. Pick $C > 0$ large so that the graphs of $X$ and $C - Y$ are disjoint.

- Identify points on the graph of $X$ if they are connected by a horizontal line which is below the graph; yields a continuum random tree (CRT)
- Same for $C - Y_t$ yields an independent CRT
- Glue the CRTs together by declaring points on the vertical lines to be equivalent

Q: What is the resulting structure? A: Sphere with a space-filling path. A peanosphere.
Surface is topologically a sphere by Moore’s theorem

Theorem (Moore 1925)

Let \( \cong \) be any topologically closed equivalence relation on the sphere \( S^2 \). Assume that each equivalence class is connected and not equal to all of \( S^2 \). Then the quotient space \( S^2 / \cong \) is homeomorphic to \( S^2 \) if and only if no equivalence class separates the sphere into two or more connected components.

- An equivalence relation is topologically closed iff for any two sequences \( (x_n) \) and \( (y_n) \) with
  - \( x_n \cong y_n \) for all \( n \)
  - \( x_n \to x \) and \( y_n \to y \)
- we have that \( x \cong y \).
STORY D:

SURFACE TREE PLUS
SURFACE TREE =
SURFACE PLUS
SELF-HITTING CURVE

independence on both sides
Gluing independent Lévy trees

Can view $SLE_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.

$X_t$ $C-Y_t$ $t$
Gluing independent Lévy trees

Can view $\text{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.

The two trees of quantum disks almost surely determine both the $\text{SLE}_{\kappa'}$ and the LQG surface on which it is drawn.

Can convert questions about $\text{SLE}_{\kappa'}$ into questions about $\kappa'_{4}$-stable processes.

Scaling limit of “exploration path” on random planar map should be $\text{SLE}_{6}$ on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the “bubbles” cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.
Gluing independent Lévy trees

Can view $\text{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.

The two trees of quantum disks almost surely determine both the $\text{SLE}_{\kappa'}$ and the LQG surface on which it is drawn.

Can convert questions about $\text{SLE}_{\kappa'}$ into questions about $\kappa'_4$-stable processes.

Scaling limit of “exploration path” on random planar map should be $\text{SLE}_6$ on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the “bubbles” cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.
Gluing independent Lévy trees

Can view $\text{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.

- The two trees of quantum disks almost surely determine both the $\text{SLE}_{\kappa'}$ and the LQG surface on which it is drawn.
Gluing independent Lévy trees

Can view \( \text{SLE}_{\kappa'} \) process, \( \kappa' \in (4, 8) \) as a gluing of two \( \frac{\kappa'}{4} \)-stable Lévy trees.

▶ The two trees of quantum disks almost surely determine both the \( \text{SLE}_{\kappa'} \) and the LQG surface on which it is drawn

▶ Can convert questions about \( \text{SLE}_{\kappa'} \) into questions about \( \frac{\kappa'}{4} \)-stable processes.
Gluing independent Lévy trees

Can view $\text{SLE}_{\kappa'}$ process, $\kappa' \in (4, 8)$ as a gluing of two $\frac{\kappa'}{4}$-stable Lévy trees.

The two trees of quantum disks almost surely determine both the $\text{SLE}_{\kappa'}$ and the LQG surface on which it is drawn.

- Can convert questions about $\text{SLE}_{\kappa'}$ into questions about $\frac{\kappa'}{4}$-stable processes.
- Scaling limit of “exploration path” on random planar map should be $\text{SLE}_6$ on a $\sqrt{8/3}$-LQG. Using welding machinery, we can understand well the “bubbles” cut out by such an exploration process. We can understand conditional law of unexplored region given what we have seen.
STORY E:
GROWTH ON SURFACE = “RESHUFFLED” CURVE ON SURFACE
Can we make sense of $\eta$-DBM on a $\gamma$-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.

Or we could try $\eta$-DBM on corresponding RPM, which one would expect to behave similarly....

Question: Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when $\gamma$ is larger (as we will see). They look like Chinese dragons.

We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) quantum Loewner evolution: $\text{QLE}(\gamma^2, \eta)$.

But first let’s look at some computer generated images (and some animations), starting with an Eden exploration.
Can we make sense of $\eta$-DBM on a $\gamma$-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.

Or we could try $\eta$-DBM on corresponding RPM, which one would expect to behave similarly....

Question:
Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when $\gamma$ is larger (as we will see). They look like Chinese dragons.

We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) quantum Loewner evolution: $\text{QLE}(\gamma^2, \eta)$.

But first let’s look at some computer generated images (and some animations), starting with an Eden exploration.
Can we make sense of $\eta$-DBM on a $\gamma$-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.

Or we could try $\eta$-DBM on corresponding RPM, which one would expect to behave similarly.

Question: Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when $\gamma$ is larger (as we will see). They look like Chinese dragons.
Can we make sense of $\eta$-DBM on a $\gamma$-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.

Or we could try $\eta$-DBM on corresponding RPM, which one would expect to behave similarly....

**Question:** Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when $\gamma$ is larger (as we will see). They look like Chinese dragons.

We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution:** $\text{QLE}(\gamma^2, \eta)$.
Can we make sense of $\eta$-DBM on a $\gamma$-LQG? We have shown how to tile an LQG surface with diadic squares of “about the same size” so we could run a DLA on this set of squares and try to take a fine mesh limit.

Or we could try $\eta$-DBM on corresponding RPM, which one would expect to behave similarly....

**Question:** Are there coral reefs, snowflakes, lichen, crystals, plants, lightning bolts, etc. whose growth rates are affected by a random medium (something like LQG)? The simulations look similar but have a bit more personality when $\gamma$ is larger (as we will see). They look like Chinese dragons.

We will ultimately want to construct a candidate for the scaling limit, which we will call (for reasons explained later) **quantum Loewner evolution**: $\text{QLE}(\gamma^2, \eta)$.

But first let’s look at some computer generated images (and some animations), starting with an Eden exploration.
Eden model on $\sqrt{8/3}$-LQG
DLA on a $\sqrt{2}$-LQG
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

**Important observations:**

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.

**Belief:**

Isotropic enough so that at large scales this is close to a ball in the graph metric.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.

Belief:
- Isotropic enough so that at large scales this is close to a ball in the graph metric.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

Important observations:
- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.

Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.

Belief:

- Isotropic enough so that at large scales this is close to a ball in the graph metric.

Jason Miller and Scott Sheffield
Universal structure
October 3, 2018
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.

Important observations:
- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.

Belief:

Isotropic enough so that at large scales this is close to a ball in the graph metric.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.
- If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.

Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$. 

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components.
- Exploration respects the Markovian structure of the map.
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
Eden model on planar map

- Random planar map, random vertex \( x \). Perform FPP from \( x \).

Important observations:

- Conditional law of map given ball at time \( n \) only depends on the boundary lengths of the outside components.
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.

Important observations:

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components. *Exploration respects the Markovian structure of the map.*
Eden model on planar map

- Random planar map, random vertex $x$. Perform FPP from $x$.

**Important observations:**

- Conditional law of map given ball at time $n$ only depends on the boundary lengths of the outside components. *Exploration respects the Markovian structure of the map.*

- If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
Eden model on planar map

- Random planar map, random vertex \( x \). Perform FPP from \( x \).

Important observations:

- Conditional law of map given ball at time \( n \) only depends on the boundary lengths of the outside components. *Exploration respects the Markovian structure of the map.*

- If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length

**Belief:** Isotropic enough so that at large scales this is close to a ball in the graph metric
**Variant:**

- Pick two edges on outer boundary of cluster
**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow

This exploration also respects the Markovian structure of the map. If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length. Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
**Variant:**

- Pick two **edges** on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

This exploration also respects the Markovian structure of the map.

If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
First passage percolation on random planar maps III

**Variant:**

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

This exploration also respects the Markovian structure of the map.

If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
**Variant:**

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

This exploration also respects the Markovian structure of the map.

If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
First passage percolation on random planar maps III

**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat
Variant:

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

This exploration also respects the Markovian structure of the map. If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length. Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
First passage percolation on random planar maps III

**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. \(\frac{1}{2}\)
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

This exploration also respects the Markovian structure of the map.

If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. \( \frac{1}{2} \)
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

*This exploration also respects the Markovian structure of the map.*
**Variant:**

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

*This exploration also respects the Markovian structure of the map.*

If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
**Variant:**

- Pick two *edges* on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

This exploration also respects the Markovian structure of the map.

If we work on an “infinite” planar map, the conditional law of the map in the unbounded component only depends on the boundary length.

Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball.
Continuum limit ansatz

- Sample a random planar map
Continuum limit ansatz

Sample a random planar map and two edges uniformly at random

- Sample a random planar map and two edges uniformly at random
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2 and draw percolation interface
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2 and draw percolation interface
- Conformally map to the sphere
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface
- Conformally map to the sphere

**Ansatz** Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE$_6$. 
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
- Resample the tip according to boundary length
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
- Resample the tip according to boundary length
- Repeat
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
- Resample the tip according to boundary length
- Repeat
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of $\text{SLE}_6$
- Resample the tip according to boundary length
- Repeat
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
- Resample the tip according to boundary length
- Repeat
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
- Resample the tip according to boundary length
- Repeat
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of $\text{SLE}_6$
- Resample the tip according to boundary length
- Repeat
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of $\text{SLE}_6$
- Resample the tip according to boundary length
- Repeat
- Know the conditional law of the LQG surface at each stage, using exploration results
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of $\text{SLE}_6$
- Resample the tip according to boundary length
- Repeat
- Know the conditional law of the LQG surface at each stage, using exploration results

$\text{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
- Draw $\delta$ units of SLE$_6$
- Resample the tip according to boundary length
- Repeat
- Know the conditional law of the LQG surface at each stage, using exploration results

QLE($8/3, 0$) is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.

QLE($8/3, 0$) is SLE$_6$ with **tip re-randomization**. It can be understood as a “reshuffling” of the exploration procedure associated to the peanosphere.
What is QLE($\gamma^2, \eta$)?

QLE(8/3, 0) is a member of a two-parameter family of processes called QLE($\gamma^2, \eta$)

- $\gamma$ is the type of LQG surface on which the process grows
- $\eta$ determines the manner in which it grows
What is $\text{QLE}(\gamma^2, \eta)$?

$\text{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\text{QLE}(\gamma^2, \eta)$

- $\gamma$ is the type of LQG surface on which the process grows
- $\eta$ determines the manner in which it grows

Let $\mu_{\text{HARM}}$ (resp. $\mu_{\text{LEN}}$) be harmonic (resp. length) measure on a $\gamma$-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$
\left( \frac{d\mu_{\text{HARM}}}{d\mu_{\text{LEN}}} \right)^\eta d\mu_{\text{LEN}}.
$$
What is $\text{QLE}(\gamma^2, \eta)$?

$\text{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\text{QLE}(\gamma^2, \eta)$

- $\gamma$ is the type of LQG surface on which the process grows
- $\eta$ determines the manner in which it grows

Let $\mu_{\text{HARM}}$ (resp. $\mu_{\text{LEN}}$) be harmonic (resp. length) measure on a $\gamma$-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

\[
\left( \frac{d\mu_{\text{HARM}}}{d\mu_{\text{LEN}}} \right)^\eta \ d\mu_{\text{LEN}}.
\]

- **First passage percolation**: $\eta = 0$
What is $\text{QLE}(\gamma^2, \eta)$?

$\text{QLE}(\gamma^2, \eta)$ is a member of a two-parameter family of processes called $\text{QLE}(\gamma^2, \eta)$

- $\gamma$ is the type of LQG surface on which the process grows
- $\eta$ determines the manner in which it grows

Let $\mu_{\text{HARM}}$ (resp. $\mu_{\text{LEN}}$) be harmonic (resp. length) measure on a $\gamma$-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left( \frac{d\mu_{\text{HARM}}}{d\mu_{\text{LEN}}} \right)^\eta \, d\mu_{\text{LEN}}.$$

- **First passage percolation:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$
What is $\text{QLE}(\gamma^2, \eta)$?

$\text{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\text{QLE}(\gamma^2, \eta)$

- $\gamma$ is the type of LQG surface on which the process grows
- $\eta$ determines the manner in which it grows

Let $\mu_{\text{HARM}}$ (resp. $\mu_{\text{LEN}}$) be harmonic (resp. length) measure on a $\gamma$-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$
\left( \frac{d\mu_{\text{HARM}}}{d\mu_{\text{LEN}}} \right)^\eta d\mu_{\text{LEN}}.
$$

- **First passage percolation:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$
- **$\eta$-dielectric breakdown model:** general values of $\eta$
Discrete approximation of $\text{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG
Discrete approximation of QLE(2, 1). DLA on a $\sqrt{2}$-LQG
QLE($\gamma^2, \eta$) processes we can construct

Each of the QLE($\gamma^2, \eta$) processes with ($\gamma^2, \eta$) on the orange curves is built from an SLE$_{\kappa}$ process using tip re-randomization.
STORY F:

BROWNIAN MAP = \sqrt{\frac{8}{3}}\text{-LIOUVILLE QUANTUM GRAVITY}
Dancing snake: a natural random walk on the space of discrete “snakes.”
1. The dancing snake has a scaling limit called the Brownian snake.
2. The $x$ and $y$ coordinates of the Brownian snake's head are two functions.
3. Each of these describes a tree (via the same construction we used to make CRT from Brownian motion).
4. Gluing these two trees together gives a random surface called the Brownian map.
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.

- A proof that when $\gamma^2 = \frac{8}{3}$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.

- A proof that, under the metric defined by QLE, Liouville quantum gravity is equivalent (as a random metric measure space) to the Brownian map.

- An understanding of a continuum analog of DLA on a random surface corresponding to $\gamma^2 = 2$. 

Jason Miller and Scott Sheffield

Universal structure

October 3, 2018
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
- A proof that when $\gamma^2 = 8/3$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
Some QLE-based results

- Existence of $\text{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
- A proof that when $\gamma^2 = 8/3$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
- A proof that, under the metric defined by QLE, Liouville quantum gravity is equivalent (as a random metric measure space) to the Brownian map.
Some QLE-based results

- Existence of $\mathrm{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a $\gamma$-LQG surface.
- A proof that when $\gamma^2 = 8/3$ and $\eta = 0$, QLE describes the growth of metric balls in Liouville quantum gravity.
- A proof that, under the metric defined by QLE, Liouville quantum gravity is equivalent (as a random metric measure space) to the Brownian map.
- An understanding of a continuum analog of DLA on a random surface corresponding to $\gamma^2 = 2$. 