Phase Diagrams for Melonic Tensor / Disordered Models

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Critical Phenomena in Statistical Mechanics and Quantum Field Theory

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Desirable features of a black hole model

A) Macroscopic space-time description [Schwarzschild - 1916; Kerr - 1963; ...]
   - Definition of the horizon [Finkelstein - 1958; ...]
   - Description of the interior [Kruskal - 1960; Penrose, Hawking - 1965, 1970; ...]
   - Entropy $S = A/4G_N$ [Bekenstein - 1972; Bardeen, Carter, Hawking - 1973; ...]
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   - Loss of time-reversal invariance
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   - Unitarity problems / Information loss paradox
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Can we study black holes starting from (B) and getting to (A) through holography?
Guiding principles

Existence of parameter \( N \)

Loss of time-reversal invariance / Unitarity problems

\[ \downarrow \]

Thermodynamical irreversibility (limit \( N \to \infty \))

Chaotic dynamics

\[ F_\beta(t) \sim \langle \hat{O}(0)\hat{O}(t)\hat{O}(0)\hat{O}(t) \rangle_{\beta,\text{con.}} \propto e^{\lambda_L t} \]

Lyapunov exponent saturates bound for black holes

\[ \lambda_L \leq \frac{2\pi}{\beta} \]

[ Maldacena, Shenker, Stanford - 2015 ]
SYK model \([\text{Sachdev, Ye - 1993; Kitaev - 2015}]\)

\(N\) Majorana fermions \(\psi_1, \ldots, \psi_N\) in 0 + 1 dim. with Hamiltonian

\[
H = \sum_{i<j<k<l} J_{ijkl} \psi_i \psi_j \psi_k \psi_l
\]

Quenched disorder

\[
\langle \cdot \rangle \equiv \int dJ_{ijkl} p(J_{ijkl}) \langle \cdot \rangle_{J_{ijkl}} \quad \text{with} \quad \sigma^2(J_{ijkl}) \propto J^2
\]

Some nice features

- Approximate conformal symmetry in IR \(\Rightarrow\) NAdS\(_2/\text{NCFT}_1\)
- Analytical treatment for \(N \rightarrow \infty\)
- Explicit numerics for small \(N\) (\(|\mathcal{H}| = 2^{N/2}\))
- Saturates bound for \(\lambda_L\)
$N$ Majorana fermions $\psi_1, \ldots, \psi_N$ in $0 + 1$ dim. with Hamiltonian

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- Saturates bound for $\lambda_L$

**Not a proper Quantum Field Theory :-]**
Vector and matrix models: an overview

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Symmetry: $O(D)$, $U(N)^2$ or $U(N)$

Interactions: $(\phi^2)^k$ for $k = 1, \ldots$

Diag. scaling: $D^{V-P+\varphi} = D^{1-\ell}$

Leading: cacti diagrams (auxiliary tree-level)

Applications: cond. mat. ph., CFT, higher spin gravity

nucl. ph., QCD, string theory
New large $N$ limit [Ferrari - 2017; Ferrari, Rivasseau, Valette - 2017]

$O(d) \times U(n)^2$ model for a vector of complex matrices

Interaction vertices are

$$V_B = \text{Tr} \left( X_{\mu_1} X_{\mu_2}^\dagger \cdots X_{\mu_{2s-1}} X_{\mu_{2s}}^\dagger \right)$$

Usual scaling

$$S = nd \left( \frac{1}{2} \text{Tr} (X_{\mu} X_{\mu}^\dagger) + \sum_B t_B V_B(X_{\mu}) \right)$$
$O(d) \times U(n)^2$ model for a vector of complex matrices

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Enhance 't Hooft coupling $t_B$ for $V_B$ as

$t_B = \lambda_B d^{E(B)}$ with $E(B) \geq 0$

In the $n \to \infty$ limit

$F = \sum_{g \geq 0} n^{2-2g} F_g$

In the $d \to \infty$ limit ($g$ fixed)

$F_g = \sum_{k \geq 0} d^{1+g-k/2} F_{g,k}$
Fermionic matrices in $0 + 1$ dimensions

\[
(\psi^\dagger_\mu)^a_b = (\psi^b_{\mu a})^\dagger \\
\text{with} \quad \left\{ \psi^a_{\mu b}, (\psi^\dagger_\nu)^c_d \right\} = \frac{1}{nd} \delta_{\mu \nu} \delta^a_d \delta^c_b
\]

**Desired features**

- $U(n) \times O(d)$ invariance
- Single trace Hamiltonian
- Quadratic mass term $m \text{Tr} \left( \psi^\dagger_\mu \psi_\mu \right)$
- Quartic interactions

\[
\text{Tr} \left( \psi_\mu \psi^\dagger_\nu \psi_\mu \psi^\dagger_\nu \right) = - \text{Tr} \left( \psi^\dagger_\nu \psi_\mu \psi^\dagger_\nu \psi_\mu \right) + \frac{n}{d} , \quad \text{etc.}
\]

(Combinations $\psi_\mu \psi_\mu$ and $\psi^\dagger_\mu \psi^\dagger_\mu$ are suppressed)
Inequivalent interactions

**Crossing interactions** $\implies E(B) = 1/2$

\[
\lambda \text{Tr}(\psi_\mu^\dagger \psi_\nu^\dagger \psi_\mu \psi_\nu) \quad \lambda' \text{Tr}(\psi_\mu^\dagger \psi_\nu^\dagger \psi_\mu \psi_\nu) \quad \xi \text{Tr}(\psi_\mu^\dagger \psi_\nu \psi_\mu \psi_\nu) \quad \xi^* \text{Tr}(\psi_\mu \psi_\nu^\dagger \psi_\mu \psi_\nu)
\]

**Non-crossing interactions** $\implies E(B) = 0$

\[
\kappa \text{Tr}(\psi_\mu^\dagger \psi_\nu \psi_\nu^\dagger \psi_\mu) \quad \kappa' \text{Tr}(\psi_\mu \psi_\mu^\dagger \psi_\nu^\dagger \psi_\nu) \quad \tilde{\kappa} \text{Tr}(\psi_\mu^\dagger \psi_\nu \psi_\nu^\dagger \psi_\mu) \quad \tilde{\kappa}^* \text{Tr}(\psi_\mu \psi_\nu^\dagger \psi_\mu \psi_\nu)
\]
Leading order diagrams are generated by melonic moves

Mixed structures \((\lambda, \xi), \ldots\) are avoided if we require

\[
\langle \text{Tr}(\psi_\mu \psi_\mu) \rangle = \langle \text{Tr}(\psi_\mu^\dagger \psi_\mu^\dagger) \rangle = 0
\]
Melon trees
Two basic models

- **Charge preserving model** with symmetry $O(d) \times U(n)^2$

  $$H_Q = nd \, \text{Tr}\left( m \psi_\mu^\dagger \psi_\mu + \frac{1}{2} \lambda \sqrt{d} \psi_\mu^\dagger \psi_\nu^\dagger \psi_\mu \psi_\nu \right)$$

  $$\lambda' \, \text{Tr}(\psi_\mu^\dagger \psi_\nu^\dagger \psi_\mu \psi_\nu)$$ interaction renormalizes $\lambda \mapsto \lambda + 2\lambda'$

- **Charge violating model** with symmetry $O(d) \times U(n)$

  $$H_Q = nd \, \text{Tr}\left\{ m \psi_\mu^\dagger \psi_\mu + \frac{1}{2} \sqrt{d} \left( \xi \psi_\mu^\dagger \psi_\nu \psi_\mu \psi_\nu + \xi^* \psi_\mu^\dagger \psi_\nu^\dagger \psi_\mu^\dagger \psi_\nu \right) \right\}$$

Melonic-dominated models $\Rightarrow$ Physics similar to SYK
\[ \psi^a_{\mu b} \text{ has } d \times n^2 \text{ fermionic degrees of freedom} \]

\[ \Downarrow \]

Hilbert space is \( 2^{dn^2} \) dimensional :-(

**WARNING:** Equivalence is partial and only to leading large \( N = n^2 d \) order!
Disordered model formulation

\[ \psi_{\mu_b}^a \text{ has } d \times n^2 \text{ fermionic degrees of freedom} \]

\[ \Downarrow \]

Hilbert space is \( 2^{dn^2} \) dimensional :-(

Equivalent disordered models with \( N \) Dirac fermions \( \{ \chi^i, \chi^i_j \} = \delta^i_j \)

\[ \tilde{H}_Q = m \chi^i \chi^i + \frac{\lambda_{ij}}{N^{3/2}} \chi^i \chi^j \chi^k \chi^l \]

\[ \tilde{H}_Q = m \chi^i \chi^i + \frac{\xi_{ijkl}}{N^{3/2}} \chi^i \chi^j \chi^k \chi^l + \frac{\xi_{ijkl}}{N^{3/2}} \chi^i \chi^j \chi^k \chi^l \]

Hilbert space is \( 2^N \) dimensional :-)

**WARNING:** Equivalence is partial and only to leading large \( N (= n^2 d) \) order!
Euclidean two-point function

\[ G(t) = \left\langle \text{Tr} \ T \left( \psi_\mu(t) \psi_\mu^\dagger \right) \right\rangle_\beta \]

**Fermionic perturbation theory**

- \( m \gg \lambda \implies \text{Exp. around decoupled fermionic oscillators} \)
- \( T \gg \lambda \implies \text{Non-standard (SYK-like) perturbation theory} \)

\[ G_0(t) = \frac{e^{m(\beta-t)}}{e^{m\beta} + 1} = \begin{cases} 
\frac{1}{2}\text{sign}(t) & m \to 0, \text{ then } T \to 0 \\
e^{-mt}\Theta(t) & T \to 0, \text{ then } m \to 0 
\end{cases} \]

**Feynman diagram structure \implies Schwinger-Dyson equations**

\[ \begin{array}{c}
\text{simple graph} \\
= \quad + \quad + \quad \cdots
\end{array} \]
Schwinger-Dyson equations

Expanding $G(t)$ in Matsubara-Fourier modes

$$G(t) = \frac{1}{\beta} \sum_k G_k e^{-i\omega_k t}, \quad \omega_k = \frac{2\pi}{\beta} k \quad \text{with} \quad k \in \mathbb{Z} + \frac{1}{2}$$

The Schwinger-Dyson equations are

$$G_k^{-1} = m - i\omega_k + \Sigma_k \begin{cases} 
\Sigma_Q(t) &= \lambda^2 G(t)^2 G(-t) \\
\Sigma_{\bar{Q}}(t) &= -\frac{1}{4} |\xi|^2 G(t) [G(t)^2 + 3G(-t)^2] 
\end{cases}$$

Now

- Define $S_{\text{eff}}$ with Schwinger-Dyson equations as saddle-points

- Relate its on-shell value to the free energy

$$F = -\frac{1}{\beta} \log \text{Tr} \ e^{-\beta H}$$
Charge preserving model: Phase diagram structure

High $T$ pert. regime

$T \gg 1, \frac{S}{n^2 d} = \log 2 \simeq 0.69$

Perturbative regime

$m = 0$

$T \to 0$

Strong coupling regime

High $m$ pert. regime

$G(t) = e^{-mt} \Theta(t)$

(SYK)
Building the phase diagram

\[ \frac{S}{n^2 d} \]

\[ T = 0.05 \]

SYK-like solution

\[ m_c \approx 0.304 \]

Perturbative solution

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Phase Diagrams for Melonic Tensor / Disordered Models
Phase diagram: \((m, T)\) plane

- High entropy phase: \(T_c = 0.06872\)
- Low entropy phase: \(m_c = 0.3451\)
- Supercritical phase: \(\lambda = 1\)
Stringy description of gravitational collapse

\[ m \]

\[ O(d) \]
4-point functions

\[
\left\langle T \text{Tr} \left\{ \psi_\mu(t_1) \psi^\dagger_\mu(t_2) \right\} \text{Tr} \left\{ \psi_\nu(t_3) \psi^\dagger_\nu(t_4) \right\} \right\rangle_\beta = n^2 G(t_1, t_2) G(t_3, t_4) + \frac{1}{d} F(t_1, t_2, t_3, t_4)
\]

**Leading diagrams for** $F$

\[+\]

\[+\]

\[+\]

\[+\]

\[\cdots\]

\[
F = \sum_{n=0}^{\infty} K^n \ast F_0 = (1 - K)^{-1} \ast F_0 \quad \text{with} \quad F_0 = -G(t_1, t_4) G(t_3, t_2)
\]

**Rungs**

\[\]

\[\]

\[\]

\[\]
Lyapunov exponents

\[ \frac{\lambda_L}{2\pi/\beta} \]

SYK at \( q \to \infty \)

SYK for \( \beta J \gg 1 \)

- \( m = 0 \)
- \( m = 0.1 \)
- \( m = 0.2 \)
- \( m = 0.24 \)
- \( m = 0.34 \)
- \( m = 0.4 \)
- \( m = 0.5 \)
Critical behavior of the Lyapunov exponent

\[ \frac{\lambda}{2\pi/\beta} \]

\[ \alpha_- = 0.311 \]

\[ \alpha_+ = 0.401 \]
Charge violating model: Phase diagram structure

High $T$ pert. regime

$T \gg 1$, $\frac{S}{n^2 d} = \log 2 \approx 0.69$

Perturbative regime

Strong coupling regime

High $m$ pert. regime

$m = 0$

(SYK)

$G(t) = e^{-mt} \Theta(t)$
Charge violating model

\[ |\lambda| = 1 \]

Phase Diagrams for Melonic Tensor / Disordered Models

The graph shows the phase transition for different temperatures \( T \): black line for \( T = 0 \), blue line for \( T = 0.01 \), orange line for \( T = 0.05 \), and red line for \( T = 0.1 \). The transition is indicated by a point where the curves converge.
Lyapunov exponents

\[ \frac{\lambda_i}{2\pi/\beta} \]

\[ \begin{align*}
m &= 0.05 \\
m &= 0.1 \\
m &= 0.2 \\
m &= 0.3 \\
m &= 0.4 \\
m &= 0.5 \\
m &= 0.6
\end{align*} \]
Finite $N$ picture
Finite $N$ picture

$p(E)$

$F_{\rho}(t)$

-2

$10^{-4}$

$10^{-1}$

$10^0$

$10^1$

$10^2$

$10^3$

$10^4$

$10^5$

$t$

$N = 2$

$N = 3$

$N = 4$

$N = 5$

$N = 6$

$N = 7$

$N = 8$

$N = 9$

$N = 10$

$N = 11$

$N = 12$
Finite $N$ spectrum

$N = 4$

$|\lambda| = 1$
Finite $N$ spectrum

$N = 5$

$|\lambda| = 1$
Finite $N$ spectrum

$N = 6$

$|\lambda| = 1$
Finite $N$ spectrum

\[ N = 10 \]

Quantum phase transition

| $\lambda$ | $= 1$ |
Quantum critical mass: $H_Q$

$$m_c(N) \simeq 0.232 + \frac{1.03}{N} - \frac{2.510}{N^2}$$

$$\lambda = 1$$
Quantum critical mass: $H_Q$

$$m_c(N) \approx 0.371 - \frac{0.399}{N}$$

$T = 0$
$T = 0.01$
$T = 0.05$
$T = 0.1$
Outlook

1. Many generalizations: \( q \)-body interactions, bosonic models, supersymmetry, \ldots

2. Effective description à la Landau

3. Holographic picture
Outlook

1. Many generalizations: $q$-body interactions, bosonic models, supersymmetry, ... 

2. Effective description à la Landau

3. Holographic picture

Takeaway: quantum black hole playground in a computer :-(
Thanks!
Generalizations and extensions

**q-body interacting generalization of the fermionic model I**

\[
\Sigma(t) = \lambda^2 G(t)^2 G(-t) \quad \implies \quad \Sigma(t) = (-1)^{q/2} \lambda^2 G(t)^{q/2} G(-t)^{q/2}^{-1}
\]
Bosonic model phase diagram structure

\[ H_B = nd \text{Tr} \left( \frac{m^2}{2} X_\mu X_\mu + \frac{\lambda^3}{4} \sqrt{d} X_\mu X_\nu X_\mu X_\nu \right) \]

Classical regime

Strong coupling regime

\[ x^{-1} = m^2 - \frac{\lambda^6}{\beta^2} x^3 \]
Phase diagram for a bosonic model

- Unstable region
- Two solutions ($F_1 < F_2$)
- Quantum corrections
- One solution ($S_2 < 0$)
\((q_1, q_2) = (4, 8)\) domain wall

\[ m = 0; \ T = 10^{-4} \]

\[ \Delta \]

\[ \lambda_8 = 100 \]

\[ \lambda_8 = 200 \]

\(\log \omega\)