Edge universality in interacting 2d topological insulators

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Introduction: edge transport in noninteracting quantum Hall systems and time-reversal invariant systems. Bulk-edge duality.

Many-body quantum systems. Results:
- Edge transport coefficients for quantum Hall and TRI systems.
- Interacting bulk-edge correspondence, Haldane relations.

Sketch of the proof: Renormalization group and Ward identities.

Conclusions.
Introduction:
noninteracting systems
**Integer quantum Hall effect**

- **Bulk** topological order in condensed matter systems is deeply related to the emergence of **gapless** edge modes.

- **Example.** Integer quantum Hall effect [von Klitzing *et al.* ’80]
  
  $2d$ insulators exposed to strong magnetic field and in-plane electric field.
**Integer quantum Hall effect**

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- **Example.** Integer quantum Hall effect [von Klitzing et al. '80] 2d insulators exposed to strong magnetic field and in-plane electric field.

Linear response: \( J = \sigma E + o(E) \) with \( \sigma = \) conductivity matrix:

\[
\sigma = \begin{pmatrix}
0 & \frac{n}{2\pi} \\
-\frac{n}{2\pi} & 0
\end{pmatrix}, \quad n \in \mathbb{Z}.
\]
Noninteracting fermions. $H = 1$-particle Hamiltonian, on $\ell^2(\mathbb{Z}^2; \mathbb{C}^M)$. Suppose that $\sigma(H)$ is gapped, $\mu =$ Fermi level $\in \text{gap}(H)$. 
Integer quantum Hall effect: theory

- **Noninteracting fermions.** \( H = \text{1-particle Hamiltonian}, \) on \( \ell^2(\mathbb{Z}^2; \mathbb{C}^M). \) Suppose that \( \sigma(H) \) is gapped, \( \mu = \text{Fermi level} \in \text{gap}(H). \)

- For simplicity, \( H(x; y) \equiv H(x - y). \) **Bloch decomp.:** \( H = \int_{\mathbb{T}^2} dk \, \hat{H}(k) \)
  
  Let \( \hat{P}_\mu(k) = \chi(\hat{H}(k) \leq \mu) = \text{Fermi projector}. \) **Thouless et al. ’82:**

\[
\sigma_{12} = i \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2} \text{Tr}_{\mathbb{C}^M} \hat{P}_\mu(k) [\partial_{k_1} \hat{P}_\mu(k), \partial_{k_2} \hat{P}_\mu(k)] \in \frac{1}{2\pi} \mathbb{Z}
\]

\( \sigma_{12} = \text{Chern number of Bloch bundle:} \)

\[
\mathcal{E}_B = \{ (k, u) \in \mathbb{T}^2 \times \mathbb{C}^M \mid u \in \text{Ran}\hat{P}_\mu(k) \}\]
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- IQHE for general (disordered) systems:
  - Bellissard et al. ’94. $\sigma_{12} = $ Noncommutative Chern number.
  - Avron-Seiler-Simon ’94. $\sigma_{12} = $ index of a pair of projections.
  - Aizenman-Graf ’98. Strong disorder $\Rightarrow$ Hall plateaux.
Edge states in quantum Hall systems

- Halperin ’82. Hall phases must come with robust edge currents.

- Intuition. For a weak, slowly varying vector potential $A$,

\[
\frac{Z(A)}{Z(0)} = e^{i\sigma_{12} \int_\Omega A \wedge dA + \text{irr.}} \quad \text{(gap assumption)}
\]

\[
= e^{i\sigma_{12} \int_\Omega (A+d\alpha) \wedge d(A+d\alpha) + \text{irr.}} \quad \text{(gauge inv.)}
\]

\[
= \frac{Z(A)}{Z(0)} e^{i\sigma_{12} \int_{\partial\Omega} d\alpha \wedge A + \text{irr.}} \quad \text{(Stokes)}
\]

$\sigma_{12} \neq 0 \Rightarrow$ The gap assumption cannot be true!
Edge states in quantum Hall systems: more precise

- Let $H$ be a lattice Schrödinger operator on the cylinder:

![Diagram of a cylinder with labeled vertices (0, L), (L, L), (0, 0), (L, 0) and dotted lines on the edges.]

**Figure:** Dotted lines: Dirichlet boundary conditions. Identify vertical sides.
Edge states in quantum Hall systems: more precise

- Let $H$ be a lattice Schrödinger operator on the cylinder:
- Let $H_p$ the counterpart of $H$ with periodic b.c.. Hyp.: $H_p$ is gapped.
Edge states in quantum Hall systems: more precise

- Let $H$ be a lattice Schrödinger operator on the cylinder:
- Let $H_p$ the counterpart of $H$ with periodic b.c.. Hyp.: $H_p$ is gapped. $\sigma(H)$ might differ from $\sigma(H_p)$ by the presence of edge states.

Figure: $H = \int_{T^1} dk_1 \hat{H}(k_1), \hat{H}(k_1) = 1d$ Hamiltonian. Spectrum of $\hat{H}(k_1)$.
- Red curve: eigenvalue branch $\varepsilon(k_1)$, with eigenstates (edge modes)

$$\varphi_x(k_1) = e^{ik_1x_1} \xi_{x_2}(k_1), \quad \text{with} \quad \xi_{x_2}(k_1) \sim e^{-cx_2}.$$
The bulk-edge correspondence

• **Bulk-edge duality:** relation between $\sigma_{12}$ of $H_p$ and the edge states of $H$.

$$\sigma_{12} = \sum_e \frac{\omega_e}{2\pi}$$

with $\omega_e = \pm 1$ (chirality of the edge state.)

Figure: (a) : $\sigma_{12} = \frac{1}{2\pi}$,    (b) : $\sigma_{12} = -\frac{1}{2\pi}$,    (c) : $\sigma_{12} = 0$. 
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- **Rigorous results for noninteracting systems:**
  - Hatsugai, ’93: Translation invariant systems.
  - Schulz-Baldes et al. ’00: Disordered systems (with bulk gap).
  - Graf et al. ’02: Anderson localization regime.
• Quantum Hall systems are an example of topological insulators. Necessary condition for $\sigma_{12} \neq 0$: breaking of TRS (magnetic field).

• Unbroken TRS: charge transport is trivial but spin transport is possible.

Edge spin transport

- Gapped TRI model on a cylinder, Hamiltonian $H = \int_{T^1}^{\oplus} dk_1 \hat{H}(k_1)$. TRS: $\hat{H}(k_1) = \Theta^{-1} \hat{H}(-k_1) \Theta$, with $\Theta^2 = -1$, $\Theta$ antiunitary.

Figure: $\sigma(\hat{H}(k_1))$ for the Kane-Mele model. $\sigma(\hat{H}(k_1)) = \sigma(\hat{H}(-k_1))$.

- Eigenvalues at $k_1 = -k_1$ are even degenerate (Kramers degeneracy).
  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of $H$: parity of # of pairs of edge modes at $\mu$. 
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  ![Diagram](image)

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  $\Rightarrow$ (edge) $\mathbb{Z}_2$ classification of $H$: parity of # of pairs of edge modes at $\mu$.

- Bulk $\mathbb{Z}_2$ classif. is also possible (no direct connection with transport).

- Graf-P. ’13: bulk-edge duality for TRI systems.
Many-body quantum systems
Many-body quantum systems

- Interacting many-body Fermi system on $\Lambda_L \subset \mathbb{Z}^2$.

- Fock space Hamiltonian: $\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{V}$ with

  $$\mathcal{H}_0 = \sum_x \sum_y \sum_{\sigma, \sigma'} a_{x,\sigma}^+ H(x, y) a_{y,\sigma}^- , \quad \mathcal{V} = \sum_x \sum_y v(x - y) a_{x,\sigma}^+ a_{y,\sigma'}^+ a_{y,\sigma}^- a_{x,\sigma}^-$$

  with $\{a_{x,\sigma}^+, a_{y,\sigma'}^-\} = \delta_{x,y} \delta_{\sigma,\sigma'}$, $\{a_{x,\sigma}^+, a_{y,\sigma'}^+\} = \{a_{x,\sigma}^-, a_{y,\sigma'}^-\} = 0$,

  and $H, v$ short ranged.

- Finite volume, finite temperature Gibbs state:

  $$\langle \cdot \rangle_{\beta,L} = \frac{\text{Tr} \cdot e^{-\beta(\mathcal{H} - \mu \mathcal{N})}}{\mathcal{Z}_{\beta,L}} , \quad \mathcal{Z}_{\beta,L} = \text{Tr} e^{-\beta(\mathcal{H} - \mu \mathcal{N})} , \quad \beta = 1/T$$

  with $\mu =$ chemical potential and:

  $$\mathcal{N} = \sum_x \sum_{\sigma} a_{x,\sigma}^+ a_{x,\sigma}^- \equiv \sum_x n_x .$$
Interacting bulk transport

- Periodic boundary conditions. Many-body Kubo formula:

\[
\sigma_{ij} = \lim_{\eta \to 0^+} \lim_{\beta, L \to \infty} \frac{i}{\eta L^2} \left( \int_{-\infty}^{0} dt \, e^{\eta t} \langle [J_i(t), J_j] \rangle_{\beta,L} - \langle [J_i, X_j] \rangle_{\beta,L} \right)
\]

\[X = \sum_x x n_x, \quad J = i[H, X] = \text{current operator and } J_i(t) = e^{iHt} J_i e^{-iHt}.\]

- Hastings-Michalakis ’15. Quantization of \(\sigma_{12}\). (Quasi-adiabatic methods)
  Hyp.: the ground state of \(H\) is gapped.

- Giuliani-Mastropietro-P. ’16. Universality of \(\sigma_{ij}\). (RG & Ward identities)
  Hyp.: fast enough algebraic decay of corr.. E.g.: gapped ground states; graphene-like models (+Jauslin ’16: critical Haldane model).

- Bachmann-de Roeck-Fraas ’17. Validity of Kubo formula.
  Hyp.: the ground state of \(H(t)\) is gapped for all times.
Many-body quantum systems

Interacting edge transport

- **Edge transport.** Localize observables at distance \( \leq \ell \) from \( x_2 = 0 \).

(Cylindric boundary conditions & transl. inv. in direction \( x_1 \).)

- **Interesting quantities:** charge density \( n_x \) and current density \( \vec{j}_x \),

\[
n_x = a_x^+ a_x^- , \quad \partial_t n_x(t) + \text{div}_x \vec{j}_x(t) = 0 .
\]

Let: \( \hat{n}_{p_1} = \sum_{x = (x_1, x_2)} e^{i p_1 x_1} n_x \) and \( \hat{n}_{p_1}^\ell = \sum_{x_1} e^{i p_1 x_1} \sum_{x_2 \leq \ell} n_x \).
Interacting edge transport

- Edge transport. Localize observables at distance $\leq \ell$ from $x_2 = 0$.

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$$\hat{n}_{p_1} = \sum_{x=(x_1,x_2)} e^{ip_1 x_1} n_x \quad \text{and} \quad \hat{n}_{p_1}^\ell = \sum_{x_1} e^{ip_1 x_1} \sum_{x_2 \leq \ell} n_x .$$

- Spin transport (current well defined if $[\mathcal{H}, S_3] = 0$, with $S_3 = \sum_\sigma \sigma n_{x,\sigma}$):

$$n_x \rightarrow n_{x,\uparrow} - n_{x,\downarrow} , \quad j_{1,x} \rightarrow j_{1,x,\uparrow} - j_{1,x,\downarrow} .$$
Edge transport coefficients

- **Edge charge susceptibility:**

  \[
  \kappa^{\ell} (\eta, p_1) := \lim_{\beta, L \to \infty} \frac{i}{L} \int_{-\infty}^{0} dt \; e^{\eta t} \langle [\hat{n}_{p_1}(t), \hat{n}_{-p_1}^\ell] \rangle_{\beta, L}
  \]

  (response of the edge density to a density perturbation)

- **Edge charge conductance:**

  \[
  G^{\ell} (\eta, p_1) := \lim_{\beta, L \to \infty} \frac{i}{L} \int_{-\infty}^{0} dt \; e^{\eta t} \langle [\hat{n}_{p_1}(t), \hat{j}_{1,-p_1}^{\ell}] \rangle_{\beta, L}
  \]

  (response of the edge current to a density perturbation)

- **Edge Drude weight:**

  \[
  D^{\ell} (\eta, p_1) := \lim_{\beta, L \to \infty} \frac{i}{L} \int_{-\infty}^{0} dt \; e^{\eta t} \langle [\hat{j}_{1,p_1}(t), \hat{j}_{1,-p_1}^{\ell}] \rangle_{\beta, L}
  \]

  (response of the edge current to an electric field)
Effective description of the edge modes

- Effective 1d theory for a single edge mode: chiral Luttinger model.

\[ \mathcal{H}_{\chi L} = \sum_{\sigma=\uparrow,\downarrow} \int dk \, v_e k \hat{a}_{k,\sigma}^+ \hat{a}_{k,\sigma}^- + \lambda \int dpdkdk' \hat{a}_{k+p,\uparrow}^+ \hat{a}_{k',-p,\downarrow}^+ \hat{a}_{k,\downarrow}^- \hat{a}_{k',\uparrow}^- \]

- Wen ’90. Theory of interacting Hall edge currents based on \( \chi L \).
  Advantage: \( \chi L \) exactly solvable by bosonization [Mattis-Lieb ’65.]

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- Remark. Integrability is nongeneric: broken by, e.g., nonlinearities of the dispersion relation or by the bulk degrees of freedom.
Interacting edge transport: single mode edge currents


Suppose that $H$ has one edge mode per edge. Then, $\exists \lambda_0 > 0$ s.t. for $|\lambda| < \lambda_0$ the Gibbs state is analytic in $\lambda$. Moreover, the edge transport coefficients are:

\[
\begin{align*}
\kappa^\ell(\eta, p_1) &= \frac{1}{\pi |v|} \frac{vp_1}{-i\eta + vp_1} + R_{\kappa}^\ell(\eta, p_1) \\
G^\ell(\eta, p_1) &= \frac{\omega}{\pi} \frac{vp_1}{-i\eta + vp_1} + R_{G}^\ell(\eta, p_1), \quad (\omega = \text{sgn}(v)) \\
D^\ell(\eta, p_1) &= \frac{|v|}{\pi} \frac{-i\eta}{-i\eta + vp_1} + R_{D}^\ell(\eta, p_1)
\end{align*}
\]

$v \equiv v(\lambda) = \text{dressed Fermi velocity}, \quad \lim_{\ell \to \infty} \lim_{\eta, p_1 \to 0} R_{\#}^\ell(\eta, p_1) = 0.$

- The results agrees with the predictions based on bosonization: “edge states $\simeq$ noninteracting 1d Bose gas”.
- Proof based on renormalization group methods and on a rigorous comparison with $\chi L$ [Benfatto-Falco-Mastropietro ’10+].
Bulk-edge correspondence and Haldane relations

- Interacting bulk-edge duality:
  \[
  G = \lim_{\ell \to \infty} \lim_{p_1, \eta \to 0^+} G^\ell(\eta, p_1) = \frac{\omega}{\pi} = \sigma_{12}(\lambda = 0)
  \]
  \[
  \text{bulk-edge corresp.}
  \]

  The bulk-edge duality follows from bulk universality: \(\sigma_{12}(0) = \sigma_{12}(\lambda)\).

- In contrast, the Drude weight and the susceptibility are nonuniversal:
  \[
  \kappa = \lim_{\ell \to \infty} \lim_{p_1, \eta \to 0^+} \kappa^\ell(\eta, p_1) = \frac{1}{\pi |v|}
  \]
  \[
  D = \lim_{\ell \to \infty} \lim_{\eta, p_1 \to 0^+} D^\ell(\eta, p_1) = \frac{|v|}{\pi}. \quad (v \equiv v(\lambda))
  \]

  Nevertheless, they satisfy the Haldane relation:
  \[
  \frac{D}{\kappa} = v^2
  \]

  first predicted to hold for 1d systems by [Haldane '80].
Theorem (Mastropietro-P., Phys. Rev. B ’17)

Suppose that $\mathcal{H}$ is TRS, and that $H$ has one pair of edge states per edge. Also, suppose that $[\mathcal{H}, S_3] = 0$. Then, $\exists \lambda_0 > 0$ s.t. for $|\lambda| < \lambda_0$:

$$G^s = \frac{\omega}{\pi}, \quad \omega = \text{sgn}(v).$$

Moreover the charge and spin edge Drude weights and susceptibilities are:

$$\kappa^c = \frac{K}{\pi v}, \quad D^c = \frac{vK}{\pi}, \quad \kappa^s = \frac{1}{\pi vK}, \quad D^s = \frac{v}{\pi K}$$

with $K \equiv K(\lambda) = 1 + O(\lambda) \neq 1$, $v \equiv v(\lambda) = v_{\uparrow} + O(\lambda)$. Finally, the 2-point function decays with anomalous exponent $\eta = (K + K^{-1} - 2)/2$.

Remark. In the single edge mode case, $K = 1$ (no anomalous exponents.)
Sketch of the proof
(one edge mode)
Perturbation theory

- **Wick rotation.** Transport coefficients can be expressed via imaginary time correlations \((\mathbf{T} \equiv \text{time ordering})\):

\[
G^\ell(\eta, p_1) = \lim_{\beta,L \to \infty} \int_{-\beta/2}^{\beta/2} e^{-i\eta t} \frac{1}{L} \langle \mathbf{T} \hat{n}_{p_1}(-it) ; \hat{j}^\ell_{-p_1} \rangle_{\beta,L}.
\]

- Let \(A_t \equiv A(-it)\). Perturbative expansion of Euclidean correlations:

\[
\langle \mathbf{T} A_t ; B \rangle_{\beta,L} = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{[0,\beta)^n} dt_1 \ldots dt_n \langle \mathbf{T} A_t ; B ; V_{t_1} ; \ldots ; V_{t_n} \rangle_{\beta,L} \bigg|_{\lambda=0}
\]

\(\rightsquigarrow\) Expansion in terms of Feynman diagrams. Covariance \((\beta, L \to \infty)\):

\[
g(t_1, x; t_2, y) = \langle \mathbf{T} a^\dagger_{(t_1,x)} a_{(t_2,y)} \rangle \big|_{\lambda=0} = \\
\theta(t_1 - t_2)e^{(t_2-t_1)(H-\mu)} P^\perp_{\mu}(H) - \theta(t_2 - t_1)e^{(t_2-t_1)(H-\mu)} P_{\mu}(H)
\]

- **Problems.** 1) \((2n)!\) diagrams; 2) gapless modes: slow space-time decay.
Grassmann integral formulation

\[
\frac{\text{Tr } e^{-\beta \mathcal{H}}}{\text{Tr } e^{-\beta \mathcal{H}_0}} = \int \mu(d\psi) e^{V(\psi)}
\]

- \(\psi^\pm\) = Grassmann field, \(V(\psi) = "\lambda \psi^4"\), \(\mu(d\psi) = N^{-1} e^{-\langle \psi^+, g^{-1} \psi^- \rangle} d\psi\)
- \(\psi = \psi_e + \psi_b\), where \(\psi_b\) has gapped covariance \(g^{(\text{bulk})} \equiv g\chi(|H - \mu| > \delta)\).

\[
\int \mu(d\psi) e^{V(\psi)} = \int \mu_e(d\psi_e) \mu_b(d\psi_b) e^{V(\psi_e + \psi_b)} \equiv e^{F^{(b)}_{\beta, L}} \int \mu_e(d\psi_e) e^{V^{(e)}(\psi_e)}
\]
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- \( \psi^{\pm}_x = \) Grassmann field, \( V(\psi) = "\lambda \psi^4", \) \( \mu(d\psi) = N^{-1} e^{-(\psi^+, g^{-1} \psi^-)} d\psi \)
- \( \psi = \psi_e + \psi_b, \) where \( \psi_b \) has gapped covariance \( g^{(\text{bulk})} \equiv g \chi(|H - \mu| > \delta). \)

\[ \int \mu(d\psi) e^{V(\psi)} = \int \mu_e(d\psi_e) \mu_b(d\psi_b) e^{V(\psi_e + \psi_b)} \equiv e^{F_{\beta,L}} \int \mu_e(d\psi_e) e^{V(e)(\psi_e)} \]

Brydges-Battle-Federbush formula. Solution of 2n! problem.

\[ \langle \psi^{P_1}_b; \ldots; \psi^{P_n}_b \rangle_{\mu_b} = \sum_{T \in \mathcal{T}} \alpha_T \prod_{\ell \in T} g^{(\text{bulk})}_\ell \int \nu_T(dt) \det G^{(\text{bulk})}_T(t) \]

\[ T = \text{spanning tree of } \{P_i\}, \quad \# \{T\} \leq Cn!, \quad \| \det G^{(\text{bulk})}_T(t) \|_{\infty} \leq C^n, \quad |g^{(\text{bulk})}_\ell| \leq (C/\delta) e^{-c\delta \|\ell\|}. \]
**Effective 1d model: RG analysis**

\[
\int \mu_e(d\psi_e) e^{V(e)(\psi_e)} = \int \mu_{1d}(d\varphi) e^{V(e)(\varphi*\xi^{(e)})} \equiv \int \mu_{1d}(d\varphi) e^{V^{(1d)}(\varphi)}
\]

- \((\varphi*\xi^{(e)})(\omega, k_1, x_2) = \hat{\xi}^e_{x_2}(k_1)\hat{\varphi}(\omega, k_1)\) with \(\xi^{(e)} = \) edge state and:
  \[
  \langle \hat{\varphi}^-_{(\omega,k_1)} \hat{\varphi}^+_{(\omega,k_1)} \rangle = \frac{\chi_e(k_1)}{-i\omega + \varepsilon(k_1) - \mu} \approx \frac{1}{-i\omega + v(k_1 - k_F)} \quad \chi L \text{ model}
  \]

- Multiscale evaluation of the Grassmann integral:
  
  [Gawedzki, Kupiainen, Feldman, Magnen, Rivasseau, Sénéor, Lesniewski, Benfatto, Gallavotti, Mastropietro, Balaban, Knörrer, Salmhofer, Trubowitz, Brydges, Slade...]

Write \(\varphi = \sum_{h=0}^0 \varphi^{(0)}\) and integrate \(\varphi^{(h)}\) progressively:

\[
\int \mu_{1d}(d\varphi) e^{V^{(1d)}(\varphi)} = \int \mu_{h_{\beta}}(d\varphi^{(h_{\beta})}) \cdots \mu_h(d\varphi^{(h)}) e^{V^{(h)}(\varphi^{(h_{\beta})} + \cdots + \varphi^{(h)})}
\]

\(\hat{\varphi}^{(h)}_{(\omega,k_1)}\) supported for \(|\omega|^2 + |k_1 - k_F|^2 \sim 2^{2h}\), covariance \(\hat{g}^{(h)}\).
The flow of the beta function

- Goal: control the map \((\mu_h, V^{(h)}) \rightarrow (\mu_{h-1}, V^{(h-1)})\). Morally,

\[
V^{(h)}(\varphi^{(h)}) = \int dt \sum_{x_1} \lambda_h \varphi^{(h)}_{x,\uparrow} \varphi^{(h)}_{x,\uparrow} \varphi^{(h)}_{x,\downarrow} \varphi^{(h)}_{x,\downarrow} + \text{irrelevant terms}
\]

\[
\lambda_h = \lambda_{h+1} + \beta_{h+1}(\lambda_{h+1}, \ldots, \lambda_0), \quad \lambda_0 \equiv \lambda.
\]

- In general, \(|\beta_{h+1}| \leq C \max_{k \geq h} |\lambda_k|^2\). Not summable.
The flow of the beta function

- Goal: control the map \((\mu_h, V^{(h)}) \to (\mu_{h-1}, V^{(h-1)})\). Morally,

\[
V^{(h)}(\varphi^{(h)}) = \int dt \sum_{x_1} \lambda_h \varphi^{(h)}_{x,\uparrow} - \varphi^{(h)}_{x,\downarrow} + \text{irrelevant terms}
\]

\[
\lambda_h = \lambda_{h+1} + \beta_{h+1}(\lambda_{h+1}, \ldots, \lambda_0), \quad \lambda_0 \equiv \lambda.
\]

- In general, \(|\beta_{h+1}| \leq C \max_{k \geq h} |\lambda_k|^2\). Not summable.

- Crucial remark: \(\beta_{h+1} = \beta_{h+1}^{\chi L} + \delta \beta_{h+1}\), with [Falco-Mastropietro ’08]:

\[
\beta_{h+1}^{\chi L} = 0, \quad |\delta \beta_{h+1}| \leq C 2^h \max_{k \geq h} |\lambda_k|^2.
\]

Summable iteration! Analyticity of \(V^{(h\beta)}\), unif. in \(\beta, L\), follows.
Comparison with the effective 1d theory

- RG allows to express the lattice correlations via the $\chi_L$:

$$\langle T j_{\mu,(t,x)} ; j_{\nu,y} \rangle = Z_\mu(x_2)Z_\nu(y_2)\langle T n_{(t,x_1)} ; n_{y_1} \rangle \chi_L + \text{“small errors”}$$

where $|Z_\mu(x_2)| \leq Ce^{-c|x_2|}$ (from the decay of edge modes), and:

$$(FT\langle T n_{(t,x_1)} ; n_{y_1} \rangle \chi)(\omega,p_1) = -\frac{1}{2\pi v} \frac{1}{Z^2(1-\tau)} \frac{-i\omega - vp_1}{-i\omega + \tilde{v}p_1},$$

$$\tilde{v} = (\frac{1-\tau}{1+\tau})v, \quad \tau = \frac{\lambda}{2\pi v} = \text{anomaly}, \quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2).$$
Comparison with the effective 1\textit{d} theory

- RG allows to express the lattice correlations via the $\chi_L$:

$$
\langle T_j_{\mu,(t,x)} ; j_{\nu,y} \rangle = Z_{\mu}(x_2)Z_{\nu}(y_2)\langle T n_{(t,x_1)} ; n_{y_1} \rangle \chi_L + \text{“small errors”}
$$

where $|Z_{\mu}(x_2)| \leq Ce^{-c|x_2|}$ (from the decay of edge modes), and:

$$(FT\langle T n_{(t,x_1)} ; n_{y_1} \rangle \chi_L)(\omega, p_1) = -\frac{1}{2\pi v} \frac{1}{Z^2(1-\tau)} \frac{-i\omega - vp_1}{-i\omega + \tilde{v}p_1},$$

$$
\tilde{v} = \left(\frac{1-\tau}{1+\tau}\right)v, \quad \tau = \frac{\lambda}{2\pi v} = \text{anomaly}, \quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2).
$$

(i) $D(p) = -i\omega + vp_1$; (ii) the circle localizes the lines on the UV cutoff scale; (iii) the last term vanishes as the UV cutoff is removed.
Comparison with the effective 1d theory

- RG allows to express the lattice correlations via the $\chi L$:
  \[ \langle \mathbf{T} j_{\mu,(t,x)} ; j_{\nu,y} \rangle = Z_{\mu}(x_2)Z_{\nu}(y_2)\langle \mathbf{T} n_{(t,x_1)} ; n_{y_1} \rangle \chi L + \text{“small errors”} \]
  where $|Z_{\mu}(x_2)| \leq Ce^{-c|x_2|}$ (from the decay of edge modes), and:
  \[ (FT\langle \mathbf{T} n_{(t,x_1)} ; n_{y_1} \rangle \chi L)(\omega,p_1) = -\frac{1}{2\pi v} \frac{1}{Z^2(1-\tau)} \frac{-i\omega - vp_1}{-i\omega + \tilde{v}p_1} \]
  \[ \tilde{v} = \left(\frac{1-\tau}{1+\tau}\right)v, \quad \tau = \frac{\lambda}{2\pi v} = \text{anomaly}, \quad v = v_e + O(\lambda), \quad Z = 1 + O(\lambda^2). \]

- To prove universality, need to find a cancellation between $Z, \tau, v$ and the vertex renormalization:
  \[ Z_{\mu} = \sum_{x_2} Z_{\mu}(x_2) \]
  The cancellation follows from Ward identities: consequences of the continuity equation $\partial_\mu j_{\mu,x} = 0$ for the correlations.
Ward identities

- Both $\langle \cdot \rangle_{\beta,L}$ and $\langle \cdot \rangle_{\chi L}$ satisfy vertex WIs: \( \mathbf{x} = (t, x_1, x_2) = (\mathbf{x}, x_2) \)

\[
\partial_\mu \langle T_{j,\mu}^-, z ; a_{y}^- a_{x}^+ \rangle_{\beta,L} = \left[ \delta_{x,z} \langle Ta_{y}^- a_{x}^+ \rangle_{\beta,L} - \delta_{y,z} \langle Ta_{y}^- a_{x}^+ \rangle_{\beta,L} \right]

(\partial_0 + v\partial_1) \langle T_{n,z}^- ; \varphi_{y}^- \varphi_{x}^+ \rangle_{\chi L} = \frac{1}{Z(1 - \tau)} \left[ \delta_{x,z} \langle T\varphi_{y}^- \varphi_{x}^+ \rangle_{\chi L} - \delta_{y,z} \langle T\varphi_{y}^- \varphi_{x}^+ \rangle_{\chi L} \right]
\]
Ward identities

- Both $\langle \cdot \rangle_{\beta,L}$ and $\langle \cdot \rangle_{\chi L}$ satisfy vertex WIs: $(x = (t, x_1, x_2) = (x, x_2))$

$$\partial_\mu \langle T j_\mu,z ; a^- y a^+ x \rangle_{\beta,L} = [\delta_{x,z} \langle T a^- y a^+ x \rangle_{\beta,L} - \delta_{y,z} \langle T a^- y a^+ x \rangle_{\beta,L}]$$

$$(\partial_0 + v \partial_1) \langle T n_z ; \varphi^- y \varphi^+ x \rangle_{\chi L} = \frac{1}{Z(1 - \tau)} [\delta_{x,z} \langle T \varphi^- y \varphi^+ x \rangle_{\chi L} - \delta_{y,z} \langle T \varphi^- y \varphi^+ x \rangle_{\chi L}]$$

- For large space-time distances:

$$\langle T j_{\mu,(t_1,x)} ; a^-_{(t_2,y)} a^+ z \rangle_{\beta,L} \simeq Z_\mu(x_2) \xi_{y_2} \xi_{z_2} \langle T n_{(t_1,x_1)} ; \varphi^-_{(t_2,y_1)} \varphi^+ z_1 \rangle_{\chi L}$$

$$\langle T a^-_{(t,x)} ; a^+ y \rangle_{\beta,L} \simeq \xi_{x_2} \xi_{y_2} \langle T \varphi^-_{(t,x_1)} ; \varphi^+ y_1 \rangle_{\chi L}$$

- Plugging (*) in the WIs, we get relations between $Z_\mu$, $Z$, $\tau$ and $v$:

$$Z_0 = Z(1 - \tau), \quad Z_1 = Z v(1 - \tau).$$

These identities imply the universality of the edge conductance.
Conclusions

- **Today:** Edge transport coefficients for 2d topological insulators with:
  
  (i) single-mode edge currents, or
  
  (ii) one pair of counterpropagating edge modes.

Consequences: bulk-edge duality, Haldane relation.

- Based on RG, and on Ward identities for relativistic & lattice model.

- Open problems:
  
  (i) **Multi-edge states** topological insulators? (edge states scattering?)
  
  (ii) Validity of edge **linear response theory**? (already for $\lambda = 0$!)
  
  (iii) **Fractional** Quantum Hall effect...?
Thank you!