Lace expansions

Critical Phenomena in Statistical Mechanics and Quantum Field Theory

PCTS, Princeton University

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Abstract

I will explain the general principles of lace expansions, how they have been used, and some open problems related to their future.
Plan for this lecture

The applications of the lace expansion are already beautifully reviewed – see for example (Slade, Notices of AMS Oct 2002). My only part in the process that made the lace expansion well known was to get Gordon Slade interested in it. He recruited Takashi Hara and many others; they proved the results you have heard about.

But I have had experience with all kinds of expansions and today I want to talk about general themes, particularly resummations in terms of minimal graphs. Furthermore, lace expansions, unlike Mayer expansions, are convergent up to the physical critical point. What makes this possible? This lecture heads in this direction because I think there are other good resummations waiting for us to find them.

- Self-avoiding walk and a few results
- Ideas and background for the lace expansion
- Brief comments on percolation and spin models
Self-avoiding walk (SAW)

Let $\omega = (\omega_0, \ldots, \omega_n)$ be a sequence of nearest neighbours in $\mathbb{Z}^d$ with $\omega_0 = 0$ and let $\Omega_n$ be the set of all such $\omega$ with $n$ fixed.
Self-avoiding walk (SAW)

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A SAW \( \omega \) is a sequence of distinct nearest neighbours.

Give all SAW in \( \Omega_n \) equal probability.
Weakly self avoiding walk (WSAW)

Let $\lambda \in [0, 1]$. 

SAW = WSAW with $\lambda = 1$. Simple random walk is $\lambda = 0$.

Two properties (D) and (S) these models might have:

(D). As $n \to \infty$, $E_{\lambda, n} |\omega_n|^2 \sim c_n$ for some $c$.

(S). As $t \to \infty$, $t^{-1/2} \omega([tn])$ converges in law to Brownian motion.
Weakly self avoiding walk (WSAW)

Let $\lambda \in [0, 1]$.

$$\mathbb{P}_n(\omega) := c_n \prod_{1 \leq i < j \leq n} (1 - \lambda \mathbb{1}_{\omega_i = \omega_j}), \quad \omega \in \Omega_n.$$
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- (G. Slade 1997, 1998) For $d$ sufficiently large (S) holds for SAW. Convergence of f.d. distributions from lace expansion, tightness from subadditivity.

- (Hara–Slade 1992) (D) holds for SAW with $d \geq 5$. For $d = 4$ $E|\omega_n|^2 \sim cn \log \frac{1}{4} n$ is expected.

- (Clisby–Liang–Slade 2007) Enumeration via lace expansion; in 7 dimensions there are $504,552,243,465,714,026,682,387,806$ SAW with $n = 24$ steps.

- (van der Hofstad 2001) ballistic behaviour for one-dimensional WSAW.
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The last two illustrate surprising applications of the lace expansion. The first three set the pattern that recurs with different scalings in other applications. For example, one can replace walks by lattice trees or lattice animals. In these cases the hypothesis is $d \geq 8$ and the limiting process is integrated super-Brownian excursion.
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Now on to a discussion of the lace expansion itself,
Relate SAW to simple random walk

Following Mayer in a different context, expand

\[
\prod_{0 \leq i < j \leq n} (1 - \lambda \mathbb{1}_{\omega_i = \omega_j}) = \sum_{G \subseteq \{\text{all pairs}\}} \prod_{ij \in G} (-\lambda \mathbb{1}_{\omega_i = \omega_j})
\]

The right hand side has \(2 \binom{n+1}{2}\) terms of opposing signs!

Let us see how a similar situation was handled by (O. Penrose 1967) in his work on convergence of the Mayer expansion.
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Excursion into the Mayer expansion

Particles at $x_1, \ldots, x_n$ in a finite set $\Lambda$. Grand canonical partition function

$$Z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{x \in \Lambda^n} \prod_{1 \leq i < j \leq n} (1 - f_{ij}).$$
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$$= \mathbb{1}\{x_i, x_j \text{ incompatible}\}$$

By expanding the product $Z$ becomes a sum over all graphs – connected and disconnected.
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Mayer’s first theorem: $\log Z \sim \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{G \in C(n)} \sum_{x \in \Lambda^n} \prod_{ij \in G} (-f_{ij})$
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Penrose reduced the sum over $C(n)$ to a sum over the set $T(n)$ of tree graphs = minimally connected graphs.
For each $n$ choose an order on all possible edges.
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Complete graph on $n = 5$ vertices
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Orders edges for $n = 5$
Define Kruskal map \( k : C(n) \mapsto T(n) \)
Define Kruskal map $k : \mathcal{C}(n) \mapsto \mathcal{T}(n)$

For $G$ in $\mathcal{C}(n)$ pick edges in order discarding those that form a loop.
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connected graph $G$ is mapped to tree subgraph $T$
The maximal graph $M(T)$

By construction, for any tree, $k(T) = T$. 
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Given a tree $T$, add all edges such that the resulting graph $M$ is still mapped by $k$ to $T$. One can add edges in any order to reach the same $M$. 
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All graphs $G$ such that $k(G) = T$ satisfy $T \subset G \subset M$. 
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All graphs $G$ such that $k(G) = T$ satisfy $T \subset G \subset M$.

Thus $M = M(T)$ is the maximal graph such that $k(M) = T$. 
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No graph with yellow edge $3 < \max\{7, 4\}$ can map to the red tree.
The maximal graph $M = M(T)$ such that $k(M) = T$

\[ \begin{array}{cccccc}
5 & 6 & 1 & 9 & 3 & 4 \\
2 & 9 & 3 & 4 & 7 & 5 \\
6 & 3 & 1 & 9 & 4 & 10 \\
7 & 4 & 5 & 10 & 9 & 3 \\
\end{array} \]

$M$ is the red and dotted edges, i.e., all edges except the yellow edges.
No graph with yellow edge $3 < \max\{7, 4\}$ can map to the red tree.
Likewise yellow edge $2 < \max\{5, 1, 7, 4\}$. 
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No graph with yellow edge $3 < \max\{7, 4\}$ can map to the red tree.
Likewise yellow edge $2 < \max\{5, 1, 7, 4\}$.
The graphs that map to $T$ are precisely graphs that contain $T$ and any subset of the dotted lines.
Lemma:

\[
\sum_{G \in \mathcal{C}(n)} (-f)^G = \sum_{T} (-f)^T (1 - f)^{M(T)\setminus T}.
\]
Penrose resummation formula

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\( \in [0, 1] \) if \( f_{ij} \in [0, 1] \)
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This reduction from \( \mathcal{C}(n) \) to \( \mathcal{T}(n) \) easily implies that the expansion for \( \log Z \) is absolutely convergent for \( z \) small.
Let $G_{\lambda, z}(x) := \sum_{n=0}^{\infty} z^n \sum_{\omega \in \Omega_n(x)} \prod_{1 \leq i < j \leq n} (1 - \lambda \mathbb{1}_{\omega_i = \omega_j})$. 
Back to WSAW: define Greens function

Let $G_{\lambda,z}(x) := \sum_{n=0}^{\infty} z^n \sum_{\omega \in \Omega_n(x)} \prod_{1 \leq i < j \leq n} \left(1 - \lambda \mathbb{1}_{\omega_i = \omega_j}\right)$. 

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$x \in \mathbb{Z}^d$ set of simple walks with $n$ steps and $\omega_n = x$

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$\chi_{\lambda, z} := \sum_{x \in \mathbb{Z}^d} G_{\lambda, z}(x)$ is called the susceptibility.
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Let $z_c = z_c(\lambda)$ be the radius of convergence of $\chi_{\lambda,z}$.
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**Objective:** for \( d \geq 5, \lambda \) small, \( G_{\lambda, z_c(\lambda)}(x) \leq 2 G_{0, z_c(0)}(x) \)
This is called an infrared bound. Once we get it from the lace expansion other results such as

(D). As \( n \to \infty \), \( E_{\lambda, n} |\omega_n|^2 \sim cn \) for some \( c \).

are standard.
Graphical expansion for $G_{\lambda, z}(x)$

In the formula for $G_{\lambda, z}(x)$ insert

$$\prod_{0 \leq i < j \leq n} (1 - f_{ij}) = \sum_{G \in G[0,...,n]} \prod_{ij \in G} (-f_{ij})$$
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Definition: Markovian vertices have no arches over them
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{graphs on vertices 0, \ldots, n}

Definition: Markovian vertices have no arches over them

Say $G \in C(n)$ if $G$ has no Markovian points except 0.
Define $\Pi_{\lambda, z}(x)$

$$
\Pi_{\lambda, z}(x) := \sum_{n=0}^{\infty} z^n \sum_{\omega \in \Omega_n(x)} \sum_{G \in \mathcal{C}(n)} \prod_{ij \in G} (-\lambda \mathbb{1}_{\omega_i = \omega_j})
$$

which is an expansion in graphs without Markovian points whereas

$$
G_{\lambda, z}(x) = \sum_{n=0}^{\infty} z^n \sum_{\omega \in \Omega_n(x)} \sum_{G \in \mathcal{G}(n)} \prod_{ij \in G} (-\lambda \mathbb{1}_{\omega_i = \omega_j})
$$

is an expansion in all possible graphs.
Define $k : C(n) \to L(n)$
Define $k : \mathcal{C}(n) \rightarrow \mathcal{L}(n)$
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Lemma: This map $k : \mathcal{C}(n) \to \mathcal{L}(n)$ is such that

$$\sum_{G \in \mathcal{C}(n)} (f) G = \sum_{L \in \mathcal{L}(n)} (f) L (1 - f) M(T) \setminus L, f_{ij} = \lambda_{1}, \omega_{i} = \omega_{j}.$$
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$$\sum_{G \in \mathcal{C}(n)} (-f)_G = \sum_{L \in \mathcal{L}(n)} (-f)_L (1 - f)_{M(T \setminus L)} f_{ij} = \lambda_1 \omega_i = \omega_j.$$
Define $k : \mathcal{C}(n) \rightarrow \mathcal{L}(n)$

\[
\sum_{G \in \mathcal{C}(n)} (-f)^G = \sum_{L \in \mathcal{L}(n)} (-f)^L (1 - f)^{M(T)\setminus L}, \quad f_{ij} = \lambda 1_{\omega_i = \omega_j}.
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Instead of \((1 - f)^{M(T) \setminus L} \leq 1\) used in Mayer
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\[
0 \leq (1 - f)^{M(T)\setminus L} \leq \prod_{ij \in \text{dotted edges}} (1 - \lambda \mathbb{1}_{\omega_i = \omega_j})
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Instead of $(1 - f)^{M(T)\setminus L} \leq 1$ used in Mayer

$$0 \leq (1 - f)^{M(T)\setminus L} \leq \prod_{ij \in \text{dotted edges}} (1 - \lambda 1_{\omega_i = \omega_j})$$

enabling a bootstrap. $G_{\lambda,z}$ is expressed in terms of $\Pi_{\lambda,z}$ and $\Pi_{\lambda,z}$ is bounded in terms of $G$. A poor estimate on $G$ can improve when passed through this circle.
Π bounded by \( G \)

From the last Lemma and the definition of \( Π \)

\[
Π_{λ,z}(x) = \sum_{n=0}^{∞} z^n \sum_{ω ∈ Ω_n(x)} \sum_{L ∈ L(n)} (-f)^L (1 - f)^{M(T)\setminus L},
\]
Π bounded by $G$

From the last Lemma and the definition of $\Pi$

$$\Pi_{\lambda,z}(x) = \sum_{n=0}^{\infty} z^n \sum_{\omega \in \Omega_n(x)} \sum_{L \in \mathcal{L}(n)} (-f)^L (1 - f)^{M(T) \setminus L},$$

From the $(1 - f)^{M(T) \setminus L}$ inequality, $|\Pi_{\lambda,z}(x)| \leq$

where in the Feynman diagrams on the RHS each line is $G_{\lambda,z}(\cdot)$ and each vertex has weight $\lambda$. 
Schwinger-Dyson replaces log $\leftrightarrow$ connected graphs

For $z \leq z_c$, and if $\Pi_{\lambda,z} \in \ell^1$,

$$G_{\lambda}(z) = G_0(z) + G_0(z) \ast \Pi_{\lambda}(z) \ast G_{\lambda}(z)$$
Bootstrap


If $d \geq 5$, $\lambda$ small and $z < z_c(\lambda)$

the estimate $G_{\lambda,z}(x) \leq 3G_{0,z_c(0)}(x)$

passed through the bootstrap $G_{\lambda,z} \rightarrow G_{\lambda,z} \rightarrow G_{\lambda,z}$

implies

the estimate $G_{\lambda,z}(x) \leq 2G_{0,z_c(0)}(x)$ \hspace{1cm} (3 $\Rightarrow$ 2)

For $z \ll z_c(\lambda)$, $G_{\lambda,z}(x) \leq 2G_{0,z_c(0)}(x)$ holds.

continuity properties in $z$ imply it holds $z \leq z_c(\lambda)$. 
Percolation


Whenever we expand and resum \( \prod_{1 \leq i < j \leq n} (1 - \mathbb{1}_{\omega_i = \omega_j}) \) we are developing an inclusion-exclusion formula and the percolation lace expansion is an inclusion-exclusion formula modeled on the SAW expansion. The BK inequality plays enough of the role of \( 1 - \mathbb{1}_{\omega(s) = \omega(t)} \leq 1 \) that one can get the analogue of

\[ \text{Diagram}: \quad \text{Graphs} \]

\[ + \quad + \quad + \quad + \]
Spin models

Lace expansion for the Ising model (high dimensions or finite range coupling). (A. Sakai 2007)

Application of the lace expansion to the $\phi^4$ model (A. Sakai 2015).

In preparation: similar results as Akira Sakai, but also for the two component $\phi^4$ model. (Brydges-Helmuth-Holmes)

Correlation inequalities play the role of $1 - \frac{1}{\omega(s)} = \frac{1}{\omega(t)} \leq 1$
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