Renormalization group flows in
disordered field theories

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Disorder

- In QFT we like to assume that space-time is homogeneous. But in the real world this is never true!
- Lattices have impurities, background fields (e.g. magnetic) are not constant (varying coupling constants), etc. Often ~ random.
- Can ignore if scale of variation is much larger than scale of interesting physics. Usually true in particle physics, but often not true in condensed matter physics.
Motivations for disordered RG

- Near 2\textsuperscript{nd} order phase transitions have Euclidean CFTs + disorder ("classical disorder").
- Disordered materials may have space-dependent disorder ("quantum disorder"). Can flow to QCP. (T=0)
- In pure theories we use the renormalization group, averaging over different configurations at short distances to get a useful description of the long distance physics. We want to do the same in the presence of disorder (of both types), noting that its distribution may also change with the scale.
Motivations for disordered RG

- Which fixed points can disordered field theories flow to (Euclidean, or Quantum Critical Points)? Do the disorder-averaged correlators obey a Callan-Symanzik equation?
- Even a small amount of disorder can completely change the long-distance behavior, if it couples to a relevant operator!
- Can lead to a flow to a new random fixed point with some (scale-invariant) distribution of correlation functions, or to the disappearance of the critical point altogether.
Motivations for disordered RG

- Previous studies in condensed matter physics were mostly for specific systems, and used perturbation theory (related via uncontrolled epsilon expansions to physical theories). Many results from experiments, Monte-Carlo.
- Holography: can now study also strongly coupled field theories (of specific types, with a weakly coupled + curved gravity dual). Can put in random sources, solve gravity equations, see what comes out (QCP: Hartnoll + Santos).
- We want to study most general possibilities. Understand better old results, some new ones.
Outine

• The precise setup; disorder-averaged correlation functions
• Methods: local RG and replica trick
• RG in classical disorder: mixing of couplings and of correlation functions
• RG in quantum disorder: Mixing of local and non-local correlation functions
  New critical exponent
  Emergence of Lifshitz scaling
• Summary
The Setup and Methods
Quenched disorder

• Assume that the physical state of the system does not back-react on the disorder (e.g. cause impurities to come together): it is a non-dynamical background field = quenched disorder.
• Same as random coupling constants $h(x)$.
• So, we will take an ensemble of field theories with random couplings taken from some probability distribution $P[h(x)]$, compute something for each field theory and then average over the disorder. In self-averaging situations (when variance is small) this will give the typical result. (Not always)
Simplifying assumptions

- Work in continuum limit.
- Take disorder to couple to a single scalar operator $\int d^d x \, h(x)O(x)$ or $\int d^d x \, dt \, h(x)O(x,t)$, generally most relevant operator.
- Interested in long distances compared to scale of disorder – so disorder is (very) short-range, and background fields / couplings $h(x)$ vary independently and randomly at every point. For example Gaussian:

$$P[h(x)] \propto e^{-\frac{1}{2\nu} \int d^d x \, h^2(x)}$$

$$\overline{h(x)} = 0, \quad h(x)h(y) = \nu \, \delta(x - y)$$
Precise setup

• More generally, probability distribution is some local functional

\[ P[h(x)] = e^{-\int d^d x \ p(h(x))} \]

• Disorder-averaged correlation functions defined by

\[
\langle O_1(x_1) \ldots O_n(x_n) \rangle \equiv \\
\int [Dh] P[h] \frac{\int [D\Phi] O_1(x_1) \ldots O_n(x_n) e^{-S[h]}}{\int [D\Phi] e^{-S[h]}}
\]

• Averaging over disorder restores translation inv.

• Do **not** get a standard QFT with correlators

\[
(\int [Dh] e^{-\int d^d x \ p(h(x))} \int [D\Phi] O_1(x_1) \ldots O_n(x_n) e^{-S[h]}) / Z
\]
Precise setup

• Usual definition of free energy with source:
  \[ e^{W[h]} = Z[h] = \int [D\Phi] e^{-S[h]} \]
  and then disordered free energy is
  \[ W_D = \int [Dh] W[h] e^{-\int d^d x \, p(h(x))} \]

• This governs the thermodynamical properties. Connected-disordered correlators are derivatives of this by other couplings \( g_i(x) \).

• Note: after disorder averaging connected and full correlators are independent

\[ \langle O(x_1)O(x_2) \rangle - \langle O(x_1) \rangle \langle O(x_2) \rangle \neq \langle O(x_1)O(x_2) \rangle - \langle O(x_1) \rangle \cdot \langle O(x_2) \rangle \]
Method 1: local RG

- One method to study such systems is the “local renormalization group”. Start with

\[
S[h(x)] = S_0 + \int d^d x \ h(x) O(x)
\]

and perform standard Wilsonian RG locally.

- Couplings of \( S_0 \) change, and \( h(x) \) flows to some new \( h'(x) \), so distribution \( P[h(x)] \) modified. RG flow in space of couplings + distributions. For Gaussian disorder \( \nu \) behaves like a new coupling, mixing with standard ones under RG. Generally generate disorder for other couplings, and higher moments.

- Complicated to analyze…
Method 2: replica

- A general method is replica trick: recall

\[ W_D = \int [Dh] \log(Z[h]) e^{-\frac{1}{2\nu} \int d^d x h^2(x)} = \]

\[ = \frac{d}{dn} \mid_{n=0} \int [Dh] Z^n[h] e^{-\frac{1}{2\nu} \int d^d x h^2(x)} \]

\[ Z^n[h] = \int \prod_{A=1}^{n} [D\Phi_A] e^{-\Sigma_{A=1}^{n} S_A[h(x)]]} \]

so a limit of standard field theories (\( n \) copies of original QFT all coupled to an extra non-dynamical “field” \( h(x) \)). The \( n \rightarrow 0 \) limit and the derivative can be non-trivial, but at least perturbatively (in any expansion) fine. Replica theory = integral over \( h(x) \).
Method 2: replica

• When disorder distribution is \textbf{Gaussian}, the replica theory is particularly simple: for \textit{classical disorder}

\[ S_n = \sum_{A=1}^{n} S_{0,A} - \nu \sum_{A \neq B=1}^{n} \int d^d x \, O_A(x)O_B(x) \]

(no A=B term since short-distance limit is generally singular, can be swallowed in standard couplings).

• Here it is clear that \( \nu \) behaves like a standard coupling. RG flow will generate \textit{couplings of more replicas = higher moments} of disorder distribution, and \textit{multi-replica couplings} for other operators = disordered couplings for them.
RG flows in classical disorder
Classical disorder

• In either approach get standard RG flow with new couplings. \( v \) has dimension \((d-2 \Delta_0)\) so disorder is relevant when \( \Delta_0 < d/2 \) (Harris); often all other disorder-related operators are irrelevant. In replica approach can compute perturbatively \( \beta \) for \( v \) and the other couplings, polynomials in \( n \) so no problem as \( n \rightarrow 0 \).

• The new coupling \( v \) may flow to zero and become irrelevant – end up in standard CFT – or flow to a non-zero value = a disordered CFT with statistical predictions for observables (or gapped).
Classical disorder

- Naively get a standard Callan-Symanzik (CS) equation with extra beta functions:

\[
(M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial g} + \beta_v \frac{\partial}{\partial v} + \gamma_i + \cdots) \langle O_i(x) \cdots \rangle = 0
\]

and when \( O_i \) mixes with \( O_j \) get extra terms

\[
\gamma_{ij} \langle O_j(x) \cdots \rangle
\]

- But now have new types of mixings. In the replica theory \( \sum_A O_A \) can mix with \( \sum_{A \neq B} O'_A O''_B \). In local RG such mixings arise from a mixing of \( O(x) \) with \( h(x)O'(x) \), where under disorder-averaging this \( h(x) \) contracts with \( h(y) \) from interaction term or from another mixing.
Classical disorder

- The new terms in the CS equation for $\langle O_i(x) \cdots \rangle$ look like (for example):

$$\tilde{\gamma} \left( \langle O_j(x)O_k(x) \cdots \rangle - \langle O_j(x) \rangle \langle O_k(x) \cdots \rangle \right)$$

where the second term is a novel contribution.

- At a fixed point all these correlators mix together, no simple scaling for standard (connected) correlation functions (though one term dominant in IR).
Classical disorder

• At a disordered fixed point one finds
\[
\langle O_i(x)O_i(y) \rangle_{\text{conn}} \propto \frac{1}{(x - y)^{2\Delta}},
\]
\[
\langle O_i(x)O_i(y) \rangle \propto \tilde{\gamma} \log\left(\frac{M(x - y)}{x - y}\right)/(x - y)^{2\Delta}
\]
with an anomaly in the scaling transformation = logarithmic CFT. (Gurarie, Cardy)

• In statistical mechanics most measurements involve connected correlators. But one can also measure full correlators, e.g. by scattering. The logs imply that the (normalized) variance in some scattering amplitudes grows logarithmically with the volume, so that they are not self-averaging at large volume (even for small disorder).
Disorder critical exponents

- At RG fixed points scaling behavior controlled by critical exponents $\sim$ dimensions of local operators.
- Now have an additional critical exponent $\phi$ associated with disorder. In the replica approach this is related to the anomalous dimension of the leading operator associated with the disorder

$$v \sum_{A \neq B=1}^{n} \int d^{d} x \, O_{A}(x)O_{B}(x),$$

whose anomalous dimension is independent from that of $O_{A}(x)$. (Also new subleading exponents)
RG flows in quantum disorder
Quantum disorder

• Consider now the situation where we have a quantum theory in which disorder is constant in time, e.g.

\[ S = S_0 + \int d^d x \, dt \, h(x)O(x, t) \]

\[ P[h(x)] \propto e^{-\frac{1}{2v} \int d^d x \, h^2(x)} \]

• We will see two new features:
  1) Lifshitz scaling at fixed points (Quantum Critical Points), even if start from relativistic theory.
  2) The new mixings of disorder-averaged correlators now mix local and non-local (in time) operators. This leads to new critical exponents.
Quantum disorder non-locality

- **Non-locality** naturally arises in the replica trick. When disorder distribution is Gaussian, can again perform path integral over $h(\mathbf{x})$ but now we obtain explicitly a non-local (in time) theory:

$$S_n = \sum_{A=1}^{n} S_A - \nu \sum_{A,B=1}^{n} \int d^d x \, dt \, dt' O_A(\mathbf{x}, t) O_B(\mathbf{x}, t')$$

(now we have also an $A=B$ term since the operators are not at the same point).

- Each disordered theory is local in time, but because the disorder averages involve

$$h(x, t)h(y, t') = h(x)h(y) = \nu \delta(x - y)$$

they lead to non-locality in time also in the local RG.
Quantum disorder non-locality

• In the replica theory, the non-local terms in the action lead to mixings of $O(x, t)$ with

$$O_1(x, t)(\int dt_2 O_2(x, t_2))(\int dt_3 O_3(x, t_3)) \cdots$$

• The same mixings arise in the local RG approach by mixing $O(x, t)$ with (say) $h(x)O_1(x, t)$ and then contracting with other $h(x)$’s at different times.

• In the CS equation of connected correlation functions $\langle O_i(x, t) \cdots \rangle_{\text{conn}}$ one then gets (in addition to coupling mixings) terms looking (say) like

$$\langle O_j(x, t) \left( \int dt' O_k(x, t') \right) \cdots \rangle_{\text{conn}}$$
• Recall that for classical disorder have an additional critical exponent associated with the anomalous dimension of the leading local operator associated with the disorder, given in the replica theory by

\[ \nu \sum_{A \neq B=1}^{n} \int d^d x \, O_A(x)O_B(x). \]

Its anomalous dimension is independent from that of \( O_A(x) \).
Quantum disorder critical exponent

• For quantum disorder it is usually assumed that the corresponding operator

\[ v \sum_{A,B=1}^{n} \int d^d x \, dt \, dt' \, O_A(x, t)O_B(x, t') \]

does not have an independent anomalous dimension, because the two operators

\( O_A(x, t), O_B(x, t') \)

are separated in time.

• However, related to the non-local mixings, this turns out to be wrong, as can be verified by explicit perturbative computations. It would be interesting to find this new critical exponent in experiments / simulations.
Quantum disorder Lifshitz scaling

- Generally all marginal and relevant operators will be generated by the RG flow. For quantum disorder there is always a marginal operator $T_{00}(x,t)$, (time-translation symmetry), and we expect it to be generated, namely

$$S \to S + h_{00} \int d^d x \ dt \ T_{00}(x,t)$$

$$S \to S + h_{00} \sum_{A=1}^{n} \int d^d x \ dt \ T_{00,A}(x,t)$$

- This operator can be swallowed by a rescaling of the time direction, $t \to e^{h_{00} t}$. 
Quantum disorder Lifshitz scaling

- In fact the OPE implies that it is always generated at leading order in conformal perturbation theory, in the replica theory from the limit of

\[
\nu \sum_{A,B=1}^{n} \int d^d x \, dt \, dt' O_A(x, t) O_B(x, t')
\]

where \( O_A(x, t') \) approaches \( O_B(x, t) \). This gives

\[
\beta(h_{00}) = 2 \frac{c_{\text{rot} \nu}}{c_T} + \ldots
\]

for marginal disorder, independently of any details of the theory.

- In a generic RG flow we expect that the operator could acquire a non-zero anomalous dimension .
Lifshitz scaling in Quantum disorder

- This implies that time acquires some “anomalous dimension”, and at fixed points end up with Lifshitz scale invariance with a “dynamical scaling exponent” \( z = 1 + \gamma \),

\[
 t \rightarrow \lambda^z t, \quad x \rightarrow \lambda x
\]

- At leading order \( z \) is universal and follows from the beta function on the previous slide, \( z = 1 + 2c_{\text{opt}} c_T + \cdots \)

- Such fixed points are common in non-relativistic theories, and we see here that they generically arise also in relativistic disordered theories. In fact \( z \) is an anomalous dimension just like any other! (Analysis should be relevant also in non-relativistic RG.)
Examples

• We checked all these claims in a perturbative example (5d scalars), at large $N$ (OA+Komargodski +Yankielowicz) and in a holographic model where disorder can be analyzed by solving classical gravity equations of motion (Hartnoll+Santos, Hartnoll+Ramirez+Santos).

• Holographically, large disorder can also be studied, and also flows (in some cases) to a Lifshitz fixed point, both when marginal and when relevant.
Summary
Summary

• We discussed renormalization group flows in two classes of QFTs with “quenched disorder”.
• For classical disorder have standard flow, including disorder parameters (moments of the disorder distribution). We found new types of operator mixings, leading to new types of anomalous dimensions, and to extra terms in the CS equation.
• In a special case, this agrees with Cardy’s observation of logarithms in disorder-averaged full (not necessarily connected) correlation functions.
Summary

- For quantum disorder, naturally generate Lifshitz scaling as an anomalous dimension.
- We also found mixings of local and non-local operators, and a new critical exponent.
- Some future directions:
  1) Study interesting examples!
  2) Conformal invariance? OPE? Bootstrap? Number of degrees of freedom – is there a c-theorem? Implications of replica symmetry ($Z_n$) breaking? Hyperscaling violation?
  3) Generalization to disorder independent of more dimensions is straightforward. What can one say about SYK-like theories (order $1/N$)?
The End

Thanks for listening