Frontiers of the Spinning Bootstrap

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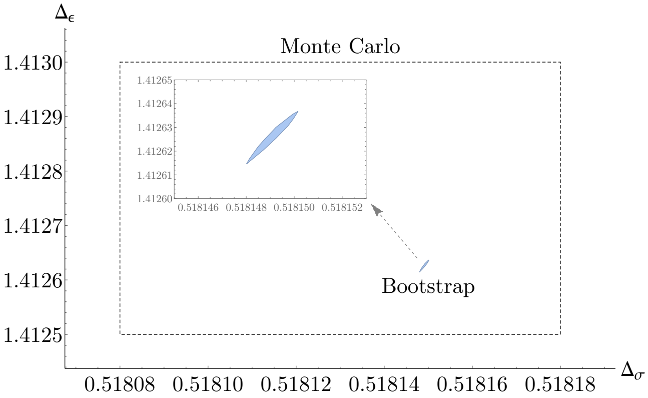
New Developments in CFT Above Two Dimensions, PCTS

3D Ising Island



[Kos, DP, Simmons-Duffin, Vichi '16]

- Many exciting results from studying bootstrap numerically, but mostly from scalar 4-point functions...

Spinning Bootstrap

There are a number of reasons to further probe spinning operators:

- All local CFTs contain stress-energy tensor $T^{\mu\nu}$
- Local CFTs with global symmetries contain currents J^{μ}
- Many interesting CFTs contain fermions ψ^{α}
- We'd like to understand role of (approximate) higher-spin currents
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4-point functions of the above operators may be needed to obtain closed islands for other interesting CFTs such as 3D QED, QCD, and Chern-Simons theories with matter, 3D Gross-Neveu, 4D QCD in conformal window, ...

Challenges for Spinning Bootstrap

Bootstrapping spinning operators comes with a few complications:

- Multiple tensor structures in 3-point functions
- Multiple tensor structures in 4-point functions
- Need to compute conformal blocks for each combination of structures
- In case of J^μ or $T^{\mu\nu}$, need to impose conservation and Ward identities

Stress Tensor

I will focus on $T^{\mu\nu}$ in 3D CFTs with parity:

- 2-point functions of $T^{\mu\nu}$ fixed by conformal symmetry in terms of a unique structure, with coefficient given by the "central charge" C:

  $$\langle TT \rangle = C \langle TT \rangle_B,$$

  where $\langle TT \rangle_B$ is the 2-point function in the theory of a free scalar field.

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- Under this normalization, $C = N$ in a theory of N free scalars or N free Majorana fermions, so it at least roughly scales with the number of degrees of freedom (however, it doesn't satisfy a C-theorem).

Stress Tensor

- 3-point functions of $T^{\mu\nu}$ contain two possible parity-preserving structures, which we can parametrize as

$$\langle TTT \rangle = n_B \langle TTT \rangle_B + n_F \langle TTT \rangle_F$$

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- Ward identities imply the relation

$$n_B + n_F = C$$

so there is one new coefficient, expressible as an angle:

$$\tan(\theta) = n_F / n_B$$

Conformal Collider Bounds

- Positivity of energy flux $\langle T|\mathcal{E}(\varphi)|T\rangle \geq 0$ at all angles φ implies that:

$$n_B \geq 0, n_F \geq 0$$

or equivalently

$$0 \leq \theta \leq \pi/2$$

[Hofman, Maldacena '08; Buchel, Escobedo, Myers, Paulos, Sinha, Smolkin '09]

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- This follows from a more general averaged null energy condition: $\int dy^- \langle T_{--}\rangle \geq 0$ in any state [Faulkner, Leigh, Parrikar, Wang '16]

Sum Rules from Lightcone Bootstrap

- This can also be derived using lightcone bootstrap methods applied to $\langle \phi T T \phi \rangle \sim G_+(z, \overline{z}) + G_-(z, \overline{z})$, where $G_{+/-}$ denote contributions from operators in $\phi \times T$ that are even/odd under parity.

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- Analytically continuing $z \to z e^{-2\pi i}$ (taking $G_{+/-} \to \hat{G}_{+/-}$), and performing an appropriate contour integral in the vicinity of $(z, \overline{z}) = (1 + \sigma, 1 + \eta\sigma)$ leads to sum rules:

$$n_{B/F} \propto \lim_{R,\eta \to 0} \eta^{-\frac{1}{2}} \int_{-R}^{R} \mathrm{Re}(G_{+/-} - \hat{G}_{+/-}) d\sigma \geq 0$$

[Hartman, Jain, Kundu '15; '16; Hofman, Li, Meltzer, DP, Rejon-Barrera '16]

(see also [Hartman, Kundu, Tajdini '16] for connection to $\mathcal{E}(\varphi)$)

Large Spin Anomalous Dimensions

- These coefficients also appear in the anomalous dimensions of large-spin "double-twist" operators $\mathcal{O}_\ell \sim \phi \partial^{\ell-2} T$

$$\gamma_\ell^+ \propto -\frac{n_B}{\ell} + \ldots, \qquad \gamma_\ell^- \propto -\frac{n_F}{\ell} + \ldots$$

  which can be interpreted in an AdS dual to mean that the gravitational binding energy of 2-particle states with large ℓ must be negative

  [Li, Meltzer, DP '15; Hofman, Li, Meltzer, DP, Rejon-Barrera '16]

Numerical Bootstrap

- These analytical results are satisfying, but we expect that much more can be learned from numerical studies of the bootstrap equations

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- These analytical results are satisfying, but we expect that much more can be learned from numerical studies of the bootstrap equations
- We have now implemented the numerical bootstrap for $\langle TTTT \rangle$ in 3D CFTs with parity [Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

Stress Tensor Bootstrap

- To set up the bootstrap, first we need to classify the operators that can appear in the $T \times T$ OPE. Using 5D embedding space formalism, it is straightforward to build the structures in $\langle TT\mathcal{O}_{\pm}^{\ell}\rangle$ compatible with conformal symmetry and conservation. The counting is:

ℓ	structures
0	$1^+ + 1^-$
2	$1^+ + 1^-$
T	2^+
$2n, n \geq 2$	$2^+ + 1^-$
$2n+1, n \geq 2$	1^-

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- Imposing parity and permutation symmetry reduces the structures to 97, which we label by a "helicity" $h_i \in \{-2, -1, 0, 1, 2\}$ at each point

$$\langle TTTT \rangle \quad = \quad \sum_{\substack{h_i/\mathbb{Z}_2^2 \\ \sum_i h_i \text{ even}}} \langle h_1 h_2 h_3 h_4 \rangle g_{[h_1 h_2 h_3 h_4]}(z, \bar{z})$$

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- Looks hopeless! However, most of the resulting 97 crossing symmetry equations are redundant, thanks to the conservation condition $\partial T = 0$.

Stress Tensor Bootstrap

- The conservation condition leads to a set of linear differential equations for the $g_i(z, \overline{z})$, which can be written in the form:

$$(\mathcal{A}\partial_z + \mathcal{B}\partial_{\overline{z}} + \mathcal{C})g = 0,$$

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- Thinking of z as a "time" variable, one can integrate this equation given some boundary data. However, this data cannot determine the structures in the kernel of $\mathcal{A}$, which has dimension 5.

  [Dymarsky '13; Kravchuk, Simmons-Duffin '16]

Stress Tensor Bootstrap

- A basis for these 5 "functional degrees of freedom" can be taken as:

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- Along this line, there are 22 possible structures invariant under $O(2)$:
  - 5 are the restrictions of the "functional degrees of freedom"
  - 9 are undetermined functions of $z=\overline{z}$,
  - 8 are unknown integration constants which we specify at $z=\overline{z}=1/2$.

Stress Tensor Bootstrap

To summarize, we have an independent crossing symmetry constraint:

$$g_{[h_1 h_2 h_3 h_4]}(z, \overline{z}) \quad = \quad g_{[h_3 h_2 h_1 h_4]}(1 - z, 1 - \overline{z})$$

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for 5 two-variable functions:

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and 8 integration constants:

$$\begin{gathered} g_{[0022]}(1/2), g_{[2002]}(1/2), g_{[01-12]}(1/2), g_{[-1102]}(1/2), \\ g_{[0-112]}(1/2), g_{[1-102]}(1/2), g_{[1-1-11]}(1/2), g_{[-1-111]}(1/2) \end{gathered}$$

Stress Tensor Bootstrap

- To calculate the conformal blocks, we express $\langle TT\mathcal{O}\rangle_a = D_a\langle\phi\phi\mathcal{O}\rangle$, which lets us relate $\int\langle TT\mathcal{O}\rangle_a\langle\tilde{\mathcal{O}}TT\rangle_b$ to scalar blocks $\int\langle\phi\phi\mathcal{O}\rangle\langle\tilde{\mathcal{O}}\phi\phi\rangle$

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- E.g. writing the crossing relations as:

$$-\vec{\alpha}\cdot\vec{F}_{\mathbb{1}} \;=\; \frac{1}{C}\vec{\alpha}\cdot\mathrm{Tr}\left[f_{2\times 2}(\theta)\vec{F}_{TTT}\right] + \vec{\alpha}\cdot\sum_{\mathcal{O}}(\ldots)$$

and imposing $-\vec{\alpha}\cdot\vec{F}_{\mathbb{1}} = 1$ and $\vec{\alpha}\cdot\sum_{\mathcal{O}}(\ldots) \geq 0$, we can obtain a θ-dependent lower bound:

$$C \geq \vec{\alpha}\cdot\mathrm{Tr}\left[f_{2\times 2}(\theta)\vec{F}_{TTT}\right]$$

Stress Tensor Bootstrap

- As usual in the numerical bootstrap, we take

$$\vec{\alpha} \sim \sum_{m+n \leq \Lambda} \vec{a}_{mn} \partial_z^m \partial_{\bar{z}}^n \Big|_{1/2,1/2}$$

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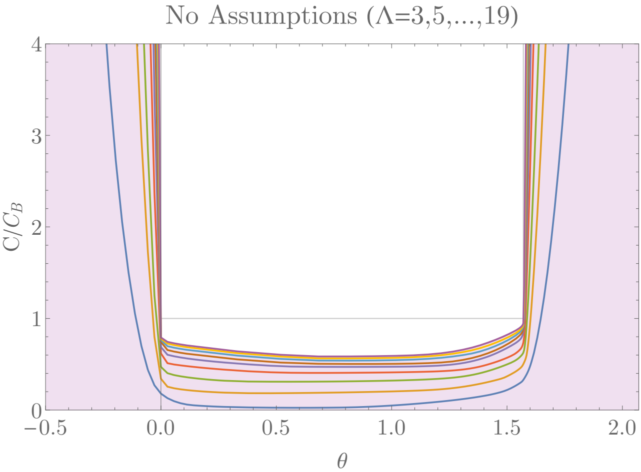
- Computations run on HPC clusters at Yale, IAS, Berkeley. Typical computation time is ~ 20 hours/point at $\Lambda = 19$.

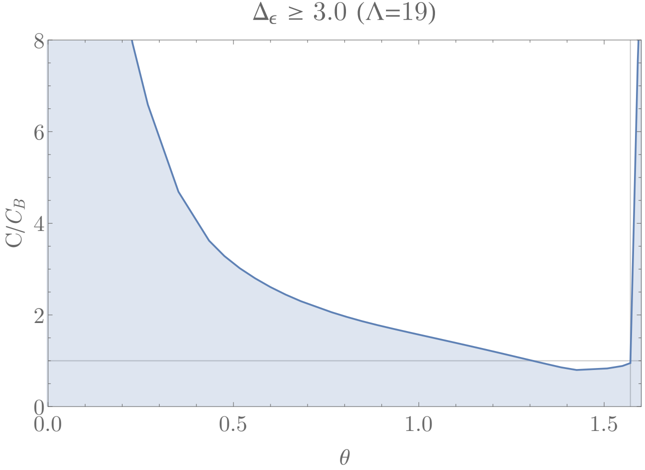
Stress Tensor Bootstrap



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Lower bound on C reproduces conformal collider bound $0 \leq \theta \leq \pi/2$

Stress Tensor Bootstrap



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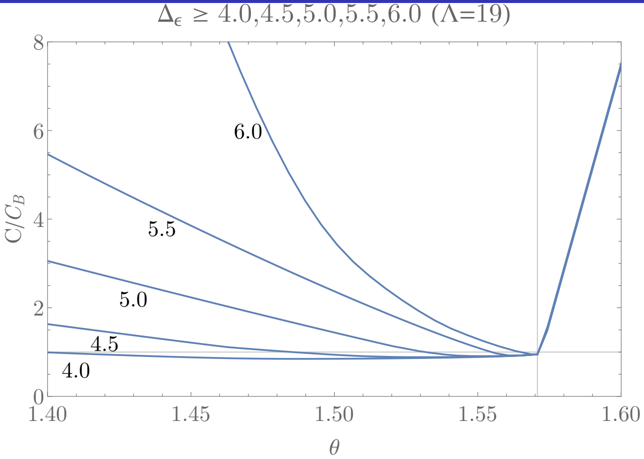
- Increasing scalar gap to $\Delta_\epsilon \geq 3$ forces $1.3 \lesssim \theta \lesssim \pi/2$ if $C/C_B \sim 1$
- Note: free scalar has gap $\Delta_\epsilon = 1$ while free fermion has gap $\Delta_\epsilon = 6$

Stress Tensor Bootstrap



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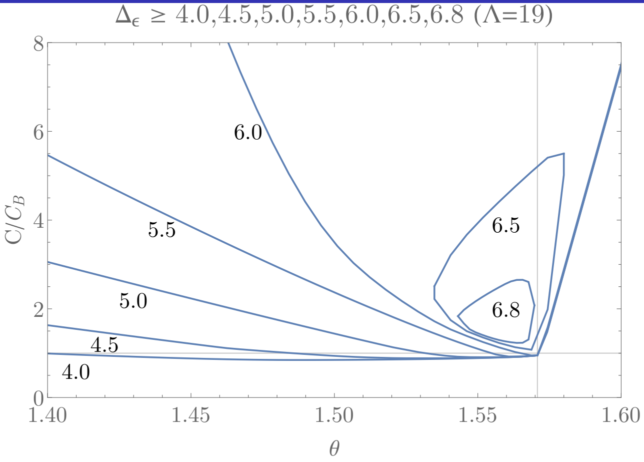
- Increasing gap assumption to $\Delta_\epsilon \geq 6$ allows only $\theta = \pi/2$ at $C/C_B = 1$
- Note: Large N theories also may have a gap of $\Delta_\epsilon = 6 + \mathcal{O}(1/N)$, where $\epsilon \simeq T_{\mu\nu}T^{\mu\nu}$ is a double-trace operator

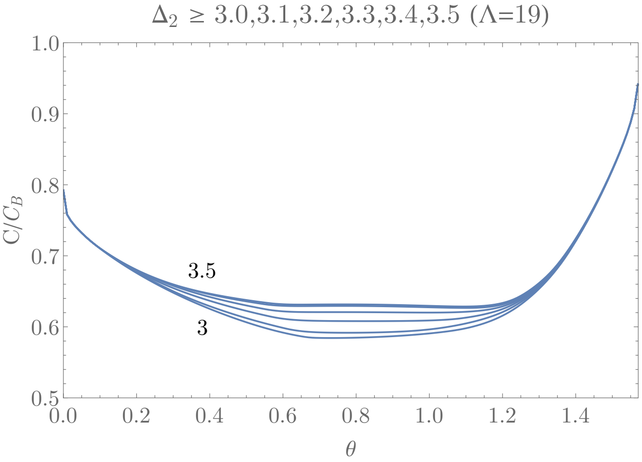
Stress Tensor Bootstrap



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Increasing our gap assumption somewhat above 6 allows us to obtain upper bounds in addition to lower bounds. The resulting closed regions force $\theta \sim \pi/2$ and $C/C_B \sim \mathcal{O}(1)$.

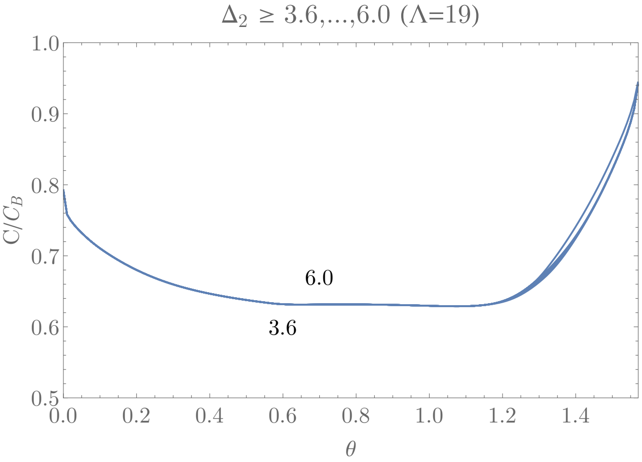
Stress Tensor Bootstrap



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Here we consider a gap in the parity-even spin-2 spectrum between the stress tensor and the next operator of dimension Δ_2.

Stress Tensor Bootstrap



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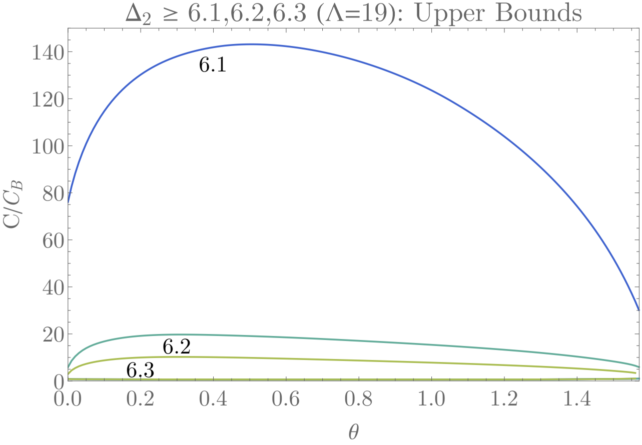
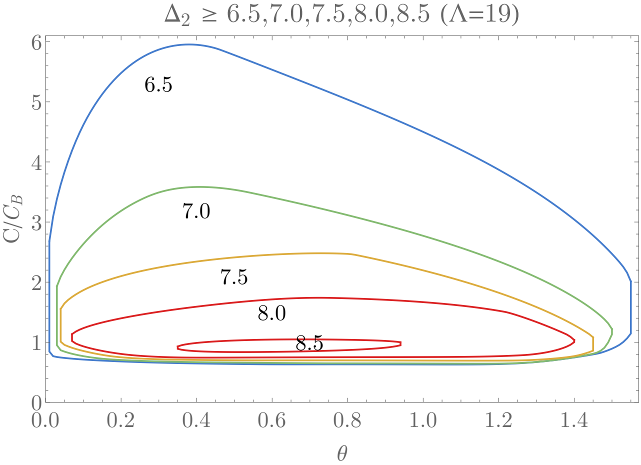


[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Once the gap is above 6, we can again obtain upper bounds. This is consistent with our expectation that large N theories should have spin 2 double-trace operators $T_{\mu\rho}T^{\rho}_{\nu}$ of dimension $\Delta_2 = 6 + \mathcal{O}(1/N)$.

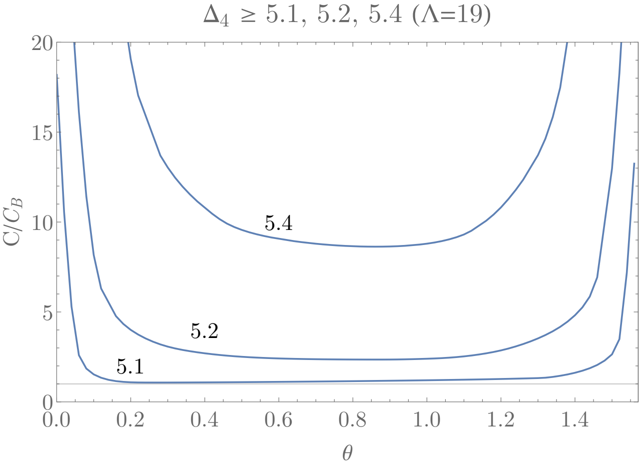
Stress Tensor Bootstrap



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Increasing the gap further, we find tightly-constrained closed regions and a universal upper bound around $\Delta_2 \lesssim 8.5$

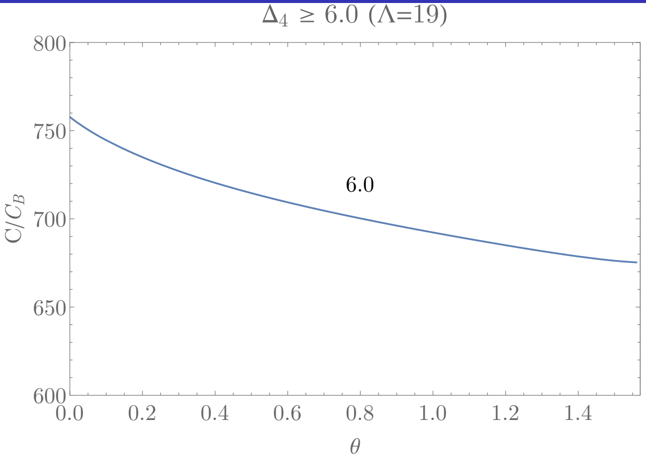
Stress Tensor Bootstrap



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Spin 4 gaps quickly force C to be large and narrows the window for θ.

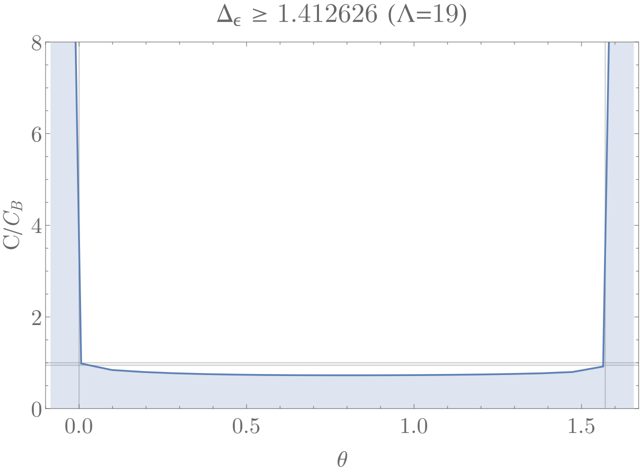
Stress Tensor Bootstrap



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- At $\Delta_4 \geq 6$, we find $C/C_B \gtrsim \mathcal{O}(700)$. Likely a hard bound $\Delta_4 \leq 6$.
- Consistent with Nachtmann theorem expectation that leading twist trajectory is increasing, concave down, and asymptotes to $\tau_{T\partial^\ell T} = 2$.

Can we learn about θ in 3d Ising?



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- Adding scalar gap in 3d Ising model doesn't significantly affect bound

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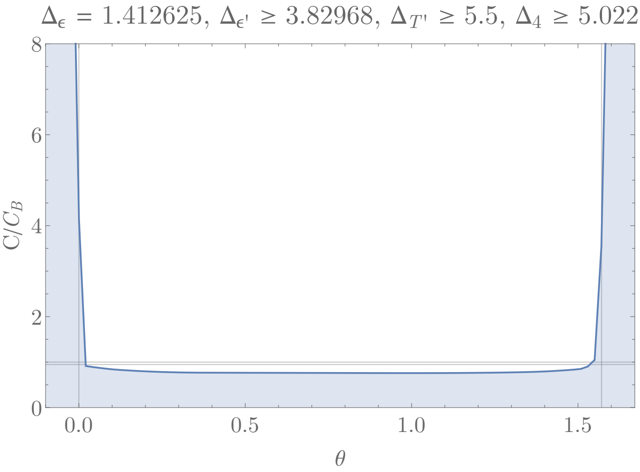
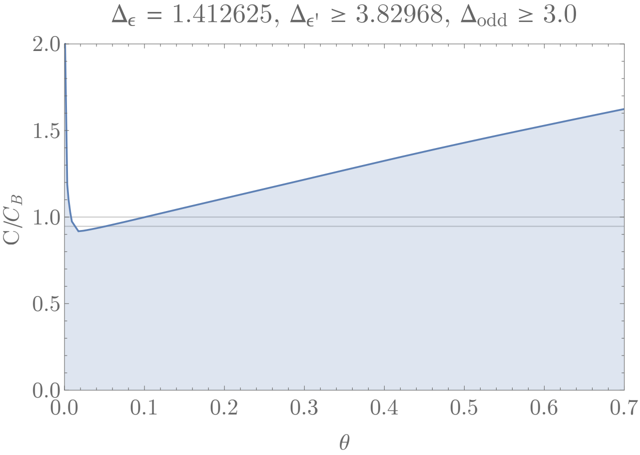


[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Adding other parity-even gaps known in 3d Ising model does not seem to help too much

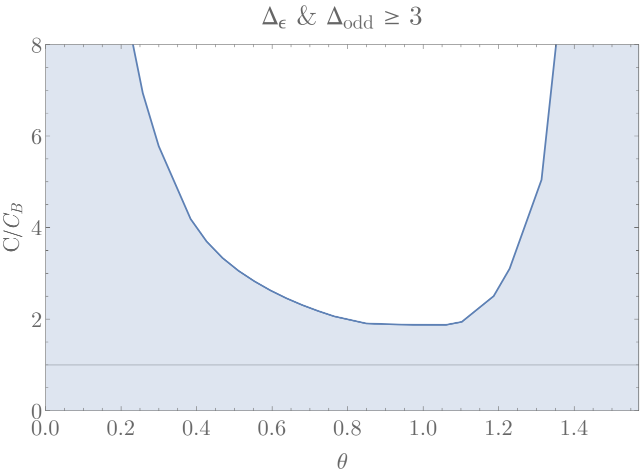
Can we learn about θ in 3d Ising?



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- However, preliminary results show that additionally imposing a parity-odd scalar gap is very constraining and forces $\theta_{\text{Ising}} \sim .01 - .04$!

Dead-end CFTs?



[Dymarsky, Kos, Kravchuk, DP, Simmons-Duffin, to appear]

- Can also start constraining space of CFTs with no relevant operators

Questions for the Future

- Can we determine precisely θ_{Ising} and $\theta(N)$ in $O(N)$ vector models?
- What other assumptions force $C \gg 1$? How does C scale as $\Delta_4 \to 6$?
- How does window for θ shrink as we go to gravity regime?
- Can we find any evidence for dead-end CFTs?
- What happens when we combine with $\langle TT\sigma\sigma\rangle$ and $\langle \sigma\sigma\sigma\sigma\rangle$?
- What happens when we allow parity-violation? Can we find e.g. Chern-Simons matter theories?