Instability of the conformal phase in QED$_3$

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LJ, Phys. Rev. D 94, 094013 (2016);
QED$_3$ historically . . . 

. . . toy model for QCD$_4$:

- asymptotic freedom
  . . . due to dimensionful fine structure constant

- chiral symmetry breaking
  . . . or conformal phase

- confinement
  . . . in its compact version

[Piarski ’84, Appelquist et al. ’88]
QED_3 today . . .

. . . effective description of high- \( T_c \)’s:

[Franz, Tesanovic, Vafek ’02; Herbut ’02; Hermele, Senthil, Fisher ’05; . . .]

. . . with \( N = 2 \) four-component Dirac fermions

. . . emergent theory of fractionalized excitations in spin systems:

[Hermele et al. ’04; Ran et al. ’07; Xu ’08; He et al. ’15; Wang & Senthil ’16; . . .]

. . . with \( N = 1, 2, 4, \ldots \)

. . . field theory of the half-filled Landau level?

[Son ’15]

. . . with \( N = 1/2 \)

→ Talk by D. T. Son, Wed 2:45pm

. . . dual description of topological insulators?

[Wang & Senthil ’15; Metlitski & Vishwanath ’15; Mross et al. ’15]

. . . with \( N = 1/2, 3/2, 5/2, \ldots \)

→ Talk by T. Senthil, Wed 11:45pm
QED$_3$: phase diagram

Possible phase diagrams of QED$_3$ with $N$ massless Dirac fermions:

[Appelquist et al. '88]
[Hands et al. '04]
[Raviv et al. '14]
[Pietro et al. '16]
[Herbut '16]

\[ \chi_{SB} \quad \text{conformal} \]
\[ N_c \quad N \]

Low $N$:
- chiral symmetry breaking ($\chi_{SB}$)
- dynamical mass generation
- Mott insulator

Large $N$:
- conformal symmetry
- interacting gapless fermions
- non-Fermi liquid
QED$_3$: phase diagram

Possible phase diagrams of QED$_3$ with $N$ massless Dirac fermions:

\[
\begin{align*}
\chi_{SB} & \quad \text{conformal} \\
N_c & \quad \text{conformal} \\
N & \quad \text{conformal}
\end{align*}
\]

\[\text{Talk by R. Narayanan, Mo 4:15pm}\]

[Appelquist et al. '88]
[Hands et al. '04]
[Raviv et al. '14]
[Pietro et al. '16]
[Herbut '16]

[...]

[Karthik & Narayanan '16]
[Chester & Pufu '16]
QED$_3$: phase diagram

Possible phase diagrams of QED$_3$ with $N$ massless Dirac fermions:

Intermediate $N$:

- spontaneous Lorentz symmetry breaking (LSB)
- gapless fermions, anisotropic propagation
- generation of charge $\langle \psi^\dagger \psi \rangle$ or current $\langle \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle$
Outline

(1) $2 + \epsilon$ expansion: $N_c^{\text{conf}} \uparrow \infty$ when $d \downarrow 2$

(2) (generalized) $F$ theorem: $N_c^{\chi_{SB}} \leq 1 + \mathcal{O}(d - 2) < N_c^{\text{conf}}$

(3) Mean-field theory & susceptibility analysis:

$$\langle v_\mu \rangle \neq 0 \text{ for } N_c^{\chi_{SB}} < N < N_c^{\text{conf}}$$
Model

Action:

\[ S_{\text{QED}} = \int d^d x \left( \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi}_i \gamma_\mu D_\mu \psi_i \right) \]

in \( 2 < d < 4 \),

where \( i = 1, \ldots, N \) and \( \mu, \nu = 0, \ldots, d - 1 \) and

- Clifford algebra: \( \{ \gamma_\mu, \gamma_\nu \} = 2\delta_{\mu\nu} \mathbb{1}_4 \) 4-dimensional
- Covariant derivative: \( D_\mu = \partial_\mu + ieA_\mu \)
- Charge: \( [e^2] = 4 - d \) RG relevant in \( d < 4 \)
- Symmetries: “chiral” \( \text{SU}(2N) \), parity, Lorentz, local \( \text{U}(1) \)
- Gauge fixing:

\[ S_{gf} = -\frac{1}{2\xi} \int d^d x (\partial_\mu A_\mu)^2, \quad \xi \in \mathbb{R} \]

allows to check gauge invariance via \( \xi \)
Renormalization group

Loop corrections can induce new operators: . . . which are not present in $S_{\text{QED}}$

Most of them irrelevant, but

local 4-fermion terms are marginal in $d = 2$!

. . . and can thus become relevant at interacting fixed points in $d = 2 + \epsilon$

Full basis:

$[\text{Gies & LJ '10}; \ldots]$

$$S_{4\text{-fermi}} = \int d^d x \left[ g_1 (\bar{\psi}_i \gamma_{35} \psi_i)^2 + g_2 (\bar{\psi}_i \gamma_\mu \psi_i)^2 \right]$$

with $\gamma_{35} \equiv i \gamma_3 \gamma_5$

. . . and where $\gamma_3$ and $\gamma_5$ the two “left-over” gamma matrices not present in $\mathcal{D}$
RG flow for $S = S_{\text{QED}} + S_{\text{gf}} + S_{\text{4-fermi}}$

Flow of charge:

$$\frac{de^2}{d\ell} = (4 - d - \eta_A)e^2 \leq 0$$

RG “time” $\ell \in [0, \infty)$

... no vertex corrections $\Leftrightarrow$ Ward identity

with anomalous dimension $\eta_A = \frac{4}{3}Ne^2 + \mathcal{O}(e^4)$.

- $e^2$ flows to strong coupling
- charged fixed point:

$$\eta_A = 4 - d$$

... exactly

... as in Abelian Higgs model: [Herbut & Tesanovic '96]
RG flow for $S = S_{\text{QED}} + S_{\text{gf}} + S_{4\text{-fermi}}$

Flow of charge:
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- $e^2$ flows to strong coupling
- charged fixed point:
  \[\eta_A = 4 - d\] ... exactly
  ... as in Abelian Higgs model: [Herbut & Tesanovic '96]

- photon propagator:
  \[D_{\mu\nu}(q) \propto |q|^{\eta_A - 2} = |q|^{2 - d} = \begin{cases} 
  \text{const.}, & d = 2 \\
  \frac{1}{|q|}, & d = 3 
\end{cases}\]
  [Schwinger '62] [Pisarski '84] [Gusynin, Hams, Reenders '01]

- strong $e^2$ generates $g_1, g_2$:
  \[
  \frac{dg_i}{d\ell} = (2 - d)g_i + \begin{array}{c}
    \text{ie} \\
    \text{ie}
  \end{array} + \begin{array}{c}
    \text{ie} \\
    \text{ie}
  \end{array} + \ldots
  \]
RG flow for $S = S_{\text{QED}} + S_{\text{gf}} + S_{4\text{-fermi}}$

Flow of charge:
$$\frac{de^2}{d\ell} = (4 - d - \eta_A)e^2 \leq 0$$

With anomalous dimension $\eta_A = \frac{4}{3}Ne^2 + \mathcal{O}(e^4)$.

- $e^2$ flows to strong coupling
- Charged fixed point: $\eta_A = 4 - d \Rightarrow e_*^2 = \frac{3(4-d)}{4N} + \mathcal{O}(1/N^2)$

...no vertex corrections $\Leftrightarrow$ Ward identity

\[ \frac{dg}{d\ell} \]

$N \to \infty$
RG flow for $S = S_{\text{QED}} + S_{gf} + S_{4\text{-fermi}}$

Flow of charge:

$$\frac{de^2}{d\ell} = (4 - d - \eta_A)e^2 \leq 0$$

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\[\text{RG “time” } \ell \in [0, \infty) \]

\[\ldots \text{no vertex corrections } \Leftrightarrow \text{Ward identity}\]
RG flow for $S = S_{\text{QED}} + S_{\text{gf}} + S_{\text{4-fermi}}$

Flow of charge:

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RG flow for $S = S_{\text{QED}} + S_{\text{gf}} + S_{\text{4-fermi}}$

Flow of charge:

$$\frac{de^2}{d\ell} = (4 - d - \eta_A)e^2 \leq 0$$

RG “time” $\ell \in [0, \infty)$

... no vertex corrections ⇔ Ward identity

with anomalous dimension $\eta_A = \frac{4}{3}Ne^2 + \mathcal{O}(e^4)$.

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\[\frac{dg}{d\ell} \quad N < N_c\]
\( 2 + \epsilon \) expansion

RG flow in \( e^2 - g_2 \) plane:

...with \( g_1 \equiv g_1^*(e^2, g_2) \) chosen such that \( \beta g_1 = 0 \)

3 interacting fixed points:

Thirring \cite{Gies & LJ,... '10, '12, '15}, conformal (cQED), quantum critical (QCP)
$2 + \epsilon$ expansion

RG flow in $e^2-g_2$ plane:

...with $g_1 \equiv g_1^*(e^2, g_2)$ chosen such that $\beta_{g_1} = 0$

![Diagram](image)

(cQED and QCP approach each other when lowering $N$)
2 + $\epsilon$ expansion

RG flow in $e^2 - g_2$ plane:

...with $g_1 \equiv g_1^*(e^2, g_2)$ chosen such that $\beta_{g_1} = 0$

$cQED$ and QCP merge when $N = N_c^{conf}$
2 + $\epsilon$ expansion

RG flow in $e^2 - g_2$ plane:

\[ g_1 \equiv g_1^*(e^2, g_2) \text{ chosen such that } \beta g_1 = 0 \]

\[
e^2 / \frac{3(4-d)}{4N}
\]

\[
g_2 / \frac{3(d-2)}{8N}
\]

red line: RG trajectory of pure QED

\[
cQED \text{ and QCP disappear for } N < N_c^{\text{conf}} \ldots
\]

\[
\ldots \text{leaving behind the runaway flow!}
\]

\[
\ldots \text{i.e., conformal phase is unstable below } N_c^{\text{conf}}
\]
### 2 + $\epsilon$ expansion

**RG flow in $e^2 - g_2$ plane:**

\[ \beta_{g_1} = 0 \]

... with \( g_1 \equiv g_1^*(e^2, g_2) \) chosen such that \( \beta_{g_1} = 0 \)

---

**Figure (d):**

- **Thirring**
- **Red line:** RG trajectory of pure QED

**Diagram:**

\[ N < N_{c}^{\text{conf}} \]

\[ N_{c}^{\text{conf}} = \frac{8 \sqrt[3]{2}}{\epsilon^{4/3}} \quad \uparrow \quad \infty \quad \text{for} \quad \epsilon \downarrow 0 \]

... agrees with [Di Pietro et al. '16]

... perturbative expansion under control
Instability of the conformal state

Common mechanism to destabilize conformal state:

- Abelian Higgs model
  - [Kaplan, Lee, Son, Stephanov '09]
- Many-flavor QCD$_4$
  - [Halperin, Lubensky, Ma '74]
- Quadratic Fermi node systems in 3D
  - [Gies & Jaeckel '06]
  - [Herbut & LJ '14, '15, '16, '17]

→ Talk by I. Herbut, Wed 2pm

Consequences:

- “Infinite-order” transition
  \[ \xi \propto \exp \left[ \frac{2\pi/\epsilon}{\sqrt{1 - (N_c/N)^{3/2}}} \right] \]
  \[ \text{...agrees with gap-equation solution: [Appelquist et al. '88]} \]
  \[ \text{...and 4 - } \epsilon \text{ expansion: [Herbut '16]} \]

- Condensates \textbf{exponentially} suppressed
  \[ \text{...numerics need large lattices for } N \lesssim N_c \]
RG monotonicity

Observation:
“effective” number of degrees of freedom decreases under RG

\[ \text{e.g., } N \text{ critical modes at Wilson-Fisher } O(N) \text{ fixed point } \rightarrow (N - 1) \text{ massless modes in the SSB phase or 0 massless modes in the symmetric phase} \]

Quantification?

1+1D: \( c \) theorem

\[ \text{[Zamolodchikov '86]} \]

\[ \boxed{c_{\text{UV}} > c_{\text{IR}}} \]

\( c \) central charge of conformal algebra

\( \ldots \) RG flow goes “downhill”
RG monotonicity

Observation:
“effective” number of degrees of freedom decreases under RG

\[ \text{e.g., } N \text{ critical modes at Wilson-Fisher } O(N) \text{ fixed point } \rightarrow (N - 1) \text{ massless modes in the SSB phase or 0 massless modes in the symmetric phase} \]

Quantification?

1+1D: \( c \) theorem

\[ c_{\text{UV}} > c_{\text{IR}} \]

\( c \) central charge of conformal algebra

[Zamolodchikov '86]

3+1D: \( a \) theorem

\[ a_{\text{UV}} > a_{\text{IR}} \]

anomaly coefficient \( a \sim \int_{S^4} \langle T_{\mu}^\mu \rangle \)

[Cardy '88; Komargodski & Schwimmer '11]

2+1D: \( F \) theorem

\[ F_{\text{UV}} > F_{\text{IR}} \]

“sphere free energy” \( F = - \log Z_{S^3} \)

[Jafferis '10; Jafferis et al. '11; Casini & Huerta '12; \ldots \]
RG monotonicity: continuous dimension?

Generalized $F$ for $d \in \mathbb{R}$:

$$\tilde{F} = \sin \left( \frac{\pi d}{2} \right) \log Z_{S_d}$$

with:

$$\tilde{F}\bigg|_{d=2} = \frac{\pi}{6} c \quad \text{c theorem} \quad \checkmark$$

$$\tilde{F}\bigg|_{d=3} = F \quad \text{F theorem} \quad \checkmark$$

$$\tilde{F}\bigg|_{d=4} = \frac{\pi}{2} a \quad \text{a theorem} \quad \checkmark$$

Thus:

$$\tilde{F}_{UV} > \tilde{F}_{IR}$$

for all integer $d$

...and various evidence for continuous $d$:

$$d = 2 + \epsilon, \quad d = 3 - \epsilon, \quad d = 4 - \epsilon, \quad d = 6 - \epsilon, \ldots$$

[Giombi, Klebanov, Tarnopolsky, Fei, ... '14, '15, '16]

...though no rigorous proof yet
**RG monotonicity: entanglement**

**Reduced density matrix:**

\[ \rho_A = \text{Tr}_\bar{A}|\psi\rangle \langle \psi| \]

**Entanglement entropy:**

\[ S_A = - \text{Tr}_A \rho_A \log \rho_A \]

E.g., \( A = \bar{A} = \{ |\uparrow\rangle, |\downarrow\rangle \} \):

\[ S_A = \begin{cases} 
0 & \text{for } |\psi\rangle = |\uparrow\uparrow\rangle \\
\log 2 & \text{for } |\psi\rangle = \frac{1}{2} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) 
\end{cases} \]

...measures entanglement between \( A \) and \( \bar{A} \)

**Scaling with subsystem size \( \sim R \)?**

- (trivial) gapped states: \( S_A \propto R^{(d-1)-1} \)
- conformal phases (or points):

\[
\begin{align*}
1+1D: & \quad S_A \propto c \log R + O(1/R) \\
2+1D: & \quad S_A \propto R - \gamma + O(1/R), \quad \text{with } \gamma = F \\
3+1D: & \quad S_A \propto R^2 + a \log R + O(1/R)
\end{align*}
\]

⇒ Universal coefficients of \( S_A \) scaling are monotonic under RG!
QED$_3$: breaking of $U(2N) \to U(N) \times U(N)$?

Compute $\tilde{F}$ in $d = 2 + \epsilon$:

**UV:**

$$\tilde{F}_{UV} = N\tilde{F}_f + (d - 2)\tilde{F}_b = N\frac{\pi}{3} + \mathcal{O}(\epsilon)$$

$\tilde{F}_f$: free fermion

$\tilde{F}_b$: free boson

**cQED:**

$$\tilde{F}_{\text{conf}} = \left( N - \frac{1}{2} \right)\tilde{F}_f = \left( 2N - 1 \right)\frac{\pi}{6} + \mathcal{O}(\epsilon)$$

$c_{\text{Schwinger}} \checkmark$

**$\chi_{SB}:$**

$$\tilde{F}_{\chi_{SB}} = \left( 2N^2 + (d - 2) \right)\tilde{F}_b = N^2\frac{\pi}{3} + \mathcal{O}(\epsilon)$$

$$\Rightarrow \quad \tilde{F}_{\chi_{SB}} > \tilde{F}_{UV} > \tilde{F}_{\text{conf}} \quad \text{for all} \ N > 1$$

Hence:

$$N_{\chi_{SB}}^c \leq 1 + \mathcal{O}(d - 2) < N_c^{\text{conf}}$$

... the phase below $N_c^{\text{conf}}$ cannot exhibit $\chi_{SB}$!

[Giombi, Klebanov, Tarnopolsky '16]

[LJ, PRD 94, 094013 (2016)]
Intermediate phase

Vafa-Witten theorem: QED$_3$ should have

(a) unbroken $U(N) \times U(N)$ symmetry, and
   ...rules out other chiral breaking patterns

(b) gapless spectrum
   ...rules out plain parity breaking

What else can it be?
Mean-field theory

Recall flow . . .

. . . towards divergent $g_2$ (and finite $e^2, \, g_1$)!

Effective description: Thirring model!

$$S_{\text{Thirring}} = \int d^d x \left[ \bar{\psi}_i \gamma_\mu \partial_\mu \psi_i + g_2 \left( \bar{\psi}_i \gamma_\mu \psi_i \right)^2 \right]$$

Large-$N$ (saddle-point) solution:

$$f_{\text{MF}}(v_\mu) = \frac{1}{2} \left( \frac{1}{g_2} - \frac{40N}{3} \right) v_\mu^2 + \sqrt{6\pi} N \left( v_\mu^2 \right)^{3/2} + \mathcal{O}(v^4)$$

for vector order parameter $v_\mu \propto \langle \bar{\psi}_i \gamma_\mu \psi_i \rangle$

$\Rightarrow$ transition towards $v_\mu \neq 0$: spontaneous Lorentz symmetry breaking!

. . . spontaneous formation of charge and/or current

. . . consistent with Vafa-Witten and $F$ theorem
Susceptibility analysis in $d = 2 + \epsilon$: bilinear terms

Add small symmetry-breaking “seeds”: ...à la magnetic field

\[ S_{\Delta} = \int d^d x \bar{\psi}_i \left[ \Delta_{\chi_{SB}} \mathbb{1}_4 + \Delta_{PSB} \gamma_{35} + i \Delta_{Kek} (\gamma_3 \cos \varphi + \gamma_5 \sin \varphi) \right. \]

\[ \left. + i \Delta^\mu_{LSB} \gamma_\mu \right] \psi_i \]

Flow of bilinears:

\[ \partial_\ell \Delta_\alpha = (1 - \eta_\psi) \Delta_\alpha + \begin{array}{c} \Delta_\alpha \leftrightarrow g_1 \\ \Delta_\alpha \leftrightarrow g_2 \end{array} + \begin{array}{c} \Delta_\alpha \leftrightarrow i e \end{array} \]

\[ \equiv x_\alpha \Delta_\alpha + \mathcal{O}(\Delta^2) \]

...with $\Delta_\alpha \in \{ \Delta_{\chi_{SB}}, \Delta_{PSB}, \Delta_{Kek}, \Delta^\mu_{LSB} \}$
Critical exponents in $d = 2 + \epsilon$

Scaling form of dimensionless free energy:

$$f(\delta \bar{g}, \Delta_\alpha) = b^{-d} f(b^y \delta \bar{g}, b^{x_\alpha} \Delta_\alpha)$$

$$\Rightarrow \quad \gamma_\alpha = \frac{2x_\alpha - d}{y} \quad \& \quad \nu = \frac{1}{y}$$

$N \lesssim N_c^{\text{conf}}$: Flow “hovers” over complex pair of fixed points QCP/cQED:

$\Rightarrow$ dominant order $\equiv$ largest $\gamma$ at QCP

$$\frac{1}{\nu} = \epsilon \sqrt{1 - (N_c^{\text{conf}}/N)^{3/2}} + \mathcal{O}(\epsilon^2),$$

$\gamma_{\chi \text{SB}} < 0, \quad \gamma_{\text{PSB}} < 0, \quad \gamma_{\text{Kek}} < 0,$

$\cdots$ superconducting gaps also lead to $\gamma_{\text{SC}} < 0$

$$\gamma_{\text{LSB}} = 1 + \mathcal{O}(\epsilon) > 0 \, !$$

$\Rightarrow$ instability towards spontaneous Lorentz symmetry breaking!
Towards $d = 3$

In $d = 3$ and finite $N$: strong-coupling problem

$N_c^{\chi_{SB}}$ controlled by $F$ theorem:

$$F_{\text{conf}} = NF_f + \frac{1}{2} \log \left( \frac{\pi N}{4} \right) + \mathcal{O}(N^{-1})$$  

$F_{\chi_{SB}} = (2N^2 + 1)F_b$

$$\Rightarrow N_c^{\chi_{SB}} \leq 4.422$$

$N_c^{\text{conf}}$ by RG flow:

naive extrapolation to $\epsilon \rightarrow 1$: $N_c^{\text{conf}} \approx 8\sqrt{2} \approx 10.1$

one-loop in fixed $d = 3$: $N_c^{\text{conf}} \approx 6.24$

functional RG: $N_c^{\text{conf}} \approx 4.1 \ldots 10.0$  

$N_c^{\chi_{SB}} < N_c^{\text{conf}}$ possibly even in $d = 3!$

...higher-order calculations necessary
Phase diagram of QED$_{2<d<4}$

\[ N \simeq \frac{8 \sqrt{2}}{\epsilon^{4/3}} \]

- \( N^{\text{conf}}_c (2 + \epsilon \text{ exp.}) \)
- \( N^{\text{conf}}_c (\text{perturb. exp.}) \)
- \( N^{\chi_{SB}}_c (F \text{ theorem}) \)

Lorentz symmetry breaking

Chiral symmetry breaking (allowed)

[JM, PRD 94, 094013 (2016)]
Conclusions

(1) QED$_3$ has **conformal** ground state at large $N > N_c^{\text{conf}} = \frac{8^{3/2}}{\epsilon^{4/3}}$ ... in $d = 2 + \epsilon$

(2) Common scenario with direct transition towards $\chi_{\text{SB}}$ inconsistent with (generalized) $F$ theorem ... at least in $d = 2 + \epsilon$ ... there are just too many Goldstones!

(3) Only possibility for $N \lesssim N_c^{\text{conf}}$: ... compatible with $F$ theorem & Vafa-Witten spontaneous **Lorentz** symmetry breaking (LSB)!

(4) Mean-field & susceptibility analyses: ... independently confirm existence of LSB phase

(5) Extrapolation $\epsilon \rightarrow 1$ & perturbative expansion in fixed $d = 3$: suggest finite window of LSB phase also in physical dimension