Fixed point collisions and tensorial order parameters in some relativistic and non-relativistic field theories

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Outline:

1) Physical motivation (from a very non-relativistic problem)

Luttinger (single-particle) Hamiltonian (for quadratic band touching)

Coulomb (1/q^2) interactions and the non-Fermi liquid fixed point near d=4

Fixed point annihilation on the way to physical dimensions d=3 and d=2

Possible instability towards tensorial (e.g. nematic) order in d=3

2) Two (relativistic) spinoffs

Chiral symmetry breaking in QED revisited

Fixed points of O(N) models near d=6
Gapless semiconductors with band inversion (gray tin, HgTe, iridates)

Luttinger (spin-orbit) Hamiltonian (p-orbitals, J = 3/2)  (Luttinger 1956)

\[
H = \frac{1}{2m} \left( (\gamma_1 + \frac{5}{2}\gamma_2)k^2 - 2\gamma_2 (\mathbf{k} \cdot \mathbf{S})^2 \right)
\]

with (rotationally symmetric) eigenvalues

\[
E_L(k) = \frac{\gamma_1 + 2\gamma_2}{2m} k^2, \quad E_H(k) = \frac{\gamma_1 - 2\gamma_2}{2m} k^2
\]
Luttinger Hamiltonian à la Dirac:

\[ H(k) = \epsilon(k) + \frac{\gamma_2}{m} d_a \Gamma^a \]

where,

\[ \epsilon(k) = \frac{\gamma_1}{2m} k^2, \quad d_a(k) = -3\xi^{ij}_a k_i k_j, \]

\[ d_1 = -\sqrt{3} k_y k_z, \quad d_2 = -\sqrt{3} k_x k_z, \quad d_3 = -\sqrt{3} k_x k_y \]

\[ d_4 = -\frac{\sqrt{3}}{2} (k_x^2 - k_y^2), \]

\[ d_5 = -\frac{1}{2} (2k_z^2 - k_x^2 - k_y^2). \]

and five 4 x 4 Dirac matrices satisfy Clifford algebra:

\[ \{\Gamma^a, \Gamma^b\} = 2\delta_{ab} \]
Without the hole band, at ``zero” (low) density:

Wigner crystal

With the hole band filled and particle band empty: the system is critical

In the RG language, the charge flows with the cutoff change:

\[
\frac{de^2}{d \ln b} = (z + 2 - d)e^2 - 4e^4
\]

(Abrikosov, ZETF 1974; Moon, Xu, Kim, Balents PRL 2013)
Below and near the upper critical dimension, $d_{up} = 4$, the flow is towards a non-Fermi liquid fixed point, with the charge at

$$e_\ast^2 = \frac{15\varepsilon}{76} + O(\varepsilon^2)$$

$$\epsilon = 4 - d$$

and the dynamical critical exponent $z < 2$:

$$z = 2 - \frac{16}{15}e^2$$

This implies power-laws in various responses, such as specific heat:

$$c_v \sim T^{d/z} \approx T^{1.7}$$

Easy way to get a NFL phase in 3D!

Or not?

\[ L = L_0 + L_a + L_\psi \]

with the free part,

\[ L_0 = \psi_i^{\dagger} \left[ \partial \tau + H_0(-i\nabla) \right] \psi_i \]

and Coulomb and short-range interactions

\[ L_a = \frac{1}{2}(\nabla a)^2 + ie a \psi_i^{\dagger} \psi_i \]

\[ L_\psi = g_1 (\psi_i^{\dagger} \psi_i)^2 + g_2 (\psi_i^{\dagger} \gamma_a \psi_i)^2 + g_3 (\psi_i^{\dagger} \gamma_{ab} \psi_i)^2 \]
The RG flow of all the couplings:

\[
\frac{d e^2}{d \ln b} = (2 + z - d - \eta_a) e^2, \\
\frac{d g_1}{d \ln b} = (z - d) g_1 - (e^2 + 2g_1)g_2 - 24g_3^2, \\
\frac{d g_2}{d \ln b} = (z - d) g_2 + \frac{4(e^2 + 2g_1)g_2}{5} - \frac{(e^2 + 2g_1)^2}{20} - \frac{37 + 16N}{5} g_2^2 + \frac{112}{5} g_2 g_3 - \frac{136}{5} g_3^2, \\
\frac{d g_3}{d \ln b} = (z - d) g_3 - \frac{1}{5} (e^2 + 2g_1) g_3 + g_2^2 - 6g_2 g_3 + \frac{4(11 - 4N)}{5} g_3^2
\]

with,

\[\eta_a = N e^2 \quad z = 2 - \frac{4}{15} e^2\]

and the “charge”

\[e^2 = \frac{2m e_{el}^2}{(4\pi \hbar^2 \varepsilon)}\]
Close to and below $d=4$ there is a (IR stable) NFL fixed point, but also a (UV stable) quantum critical point at strong interaction:

They get closer, but remain separated in the coupling space!
At some “lower critical dimension” NFL and QCP collide:

In one loop calculation, this occurs at $d_l = 3.26240$, slightly above three dimensions.
Finally, below $d$ the NFL and QCP become complex, and there is only a runaway flow left:

The system is unstable towards Mott insulator.

Scale invariance lost!
Fixed point collision and annihilation:

General number of components $N$ and dimension $d$:

Near $d=2$ the collision occurs in the perturbative regime:

$$N \geq N_c(\epsilon) = \frac{64}{25\epsilon^2} + \mathcal{O}(1/\epsilon)$$

$$\epsilon = d - 2$$
Critical number of fermions in $d=3$:

<table>
<thead>
<tr>
<th>Method</th>
<th>Reference</th>
<th>$N_c(d=3)$</th>
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<tr>
<td>$2 + \epsilon$ expansion</td>
<td>Sec. III</td>
<td>2.56</td>
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<td>RG in fixed $d = 3$</td>
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<td>$1/N$ expansion in $d = 3$</td>
<td>Ref. [18]</td>
<td>$\geq 2.6(2)$</td>
</tr>
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(Janssen & IH, PRB 2017)
Order parameter for $d < d_{\text{low}}$:

$$\chi_i = 2g_2 \langle \Psi^\dagger \gamma_i \Psi \rangle$$

Out of the five $\chi_1, \ldots, \chi_5$ not all equivalent:

1. $\chi_1 \neq 0$: $\varepsilon(\bar{p})$ gapped with minimal gap at two opposite points on equator

2. $\chi_5 < 0$: $\varepsilon(\bar{p})$ gapless with gap closing at north and south pole

3. $\chi_5 > 0$: $\varepsilon(\bar{p})$ gapped with minimal gap at entire equator

Energy $E = \int \frac{d\bar{p}}{(2\pi)^3} \varepsilon(\bar{p})$ is minimized for (3): $\chi_5 > 0$ (modulo $O(3)$)
The fate of NFL: if \( d \) is above but close to \( d=3 \), the flow becomes slow close to (complex!) NFL fixed point. The RG escape time is long:

\[
b_0 = e^{c \sqrt{d_{\text{low}} - d}} - B + O(d_{\text{low}} - d)
\]

with non-universal constants \( C \) and \( B \). There is wide crossover region of the NFL behavior within the temperature window

\[
(T_c, T_*)
\]

with the critical temperature,

\[
T_c \approx T_* b_0^{-z}
\]

Characteristic energy scale for interaction effects

\[
k_B T_* \sim \frac{e_{\text{el}}^2}{\varepsilon L_*} = \frac{\hbar^2}{2mL_*^2} = \frac{4m}{m_{\text{el}} \varepsilon^2} E_0
\]

(Sherrington & Kohn, Halperin & Rice, RMP 1968)
Some numbers (HgTe):

small mass $\frac{m}{m_{el}} \approx 1/50$

high dielectric constant $\varepsilon \approx 30$

still a reasonable $T_\ast \sim 10\,\text{K} - 100\,\text{K}$

and (maybe) a detectable $T_c \approx T_\ast / 100$
CSB in QED revisited

Schwinger-Dyson, large-N, calculation of the mass gap (Appelquist, Nash, Wijewardhana, PRL 1988):

\[ \Sigma(0) = \alpha e^{(\delta+2)} \exp \left[ \frac{-2n\pi}{(32/\pi^2 N - 1)^{1/2}} \right] \]

as the number of four-component Dirac fermions N

\[ N \rightarrow 32/\pi^2 \]

from below.

This should be understandable as a fixed point collision and annihilation.
Consider QED near four space-time dimensions with (generated) quartic terms (Di Pietro et al, PRL 2016; IH, PRD 2016)

\[ L = \bar{\Psi}_n i\gamma_\mu (\partial_\mu - ieA_\mu) \Psi_n + \sum_{a=1}^{2} g_a (\bar{\Psi}_n X_a \gamma_\mu \Psi_n)^2 + \frac{F_{\mu\nu}^2}{4} \]

with

\[
X_1 = 1 \\
X_2 = \gamma_5
\]

i.e. with additional (axial) current – (axial) current interactions.
The flow in the IR \((\Lambda \rightarrow \Lambda/b)\), one loop:

\[
\begin{align*}
\beta_1 &= (2 - d)g_1 + 4(N + 1)g_1^2 - 8g_1g_2 - 6e^2g_2, \\
\beta_2 &= (2 - d)g_2 + 2(2N - 1)g_2^2 + 4g_1g_2 \\
&\quad - 6g_1^2 - 6e^2g_1 - \frac{3}{2}e^4, \\
\beta_e &= (4 - d)e^2 + \beta_{e0}(e).
\end{align*}
\]

and the charge beta-function precisely in \(d=4\) is:

\[
\beta_{e0}(e) = -\frac{4N}{3}e^4 - 4Ne^6 + O(Ne^8, N^2e^8)
\]

(Gorishny, Kataev, Larin 1991 (four loop))
Introducing linear combinations:

\[
g_{\pm} = g_1 \pm g_2
\]

equations (almost) decouple

\[
\beta_+ = (2 - d)g_+ + 2(N - 1)g_+^2 + 2Ng_-^2 - 6g_+e^2 - \frac{3}{2}e^4,
\]

\[
\beta_- = (2 - d)g_- + 6g_-^2 + 4(N + 1)g_+g_- + 6g_-e^2 + \frac{3}{2}e^4
\]

When \( N=0 \) the first equation decouples. At zero charge:

1) Gaussian stable FP  \( g_{\pm} = 0 \)

2) Critical FP  \( g_+ = 0, \ g_- = 1/3 \)

and two more (unimportant) FPs.
Note that
\[
\sum_{a=1}^{2} g_{a} (\bar{\Psi} X_{a} \gamma_{\mu} \Psi)^2 = -g_{-} [ (\bar{\Psi} \Psi)^2 - (\bar{\Psi} \gamma_{5} \Psi)^2 ]
\]

So a large positive \( g_{-} \) indeed favors CSB.

Turning on a small charge by hand FP 1 (conformal phase) and FP 2 (critical point for CSB) approach each other.

At one loop and near \( d=4 \) the fixed points collide at
\[
e_{c}^{2} = 3 - 2\sqrt{2} = 0.17157
\]

At which
\[
g_{+} = -e_{c}^{4}/2 = -0.0147 \quad g_{-} = e_{c}^{2}/2 = 0.0857
\]

at least are reasonably small.
Equating the critical and the IR fixed point value of the charge yields

\[
\frac{4 - d}{N_c} = -\lim_{N \to 0} \frac{\beta_{e0}(e_c)}{N e_c^2}
\]

and finally

\[
N_c = \frac{3(4 - d)}{4(e_c^2 + 3e_c^4)} \approx 2.88596(4 - d) + O((4 - d)^2)
\]
O(N) fixed point above four (space-time) dimensions

Above four dimensions Wilson-Fisher fixed point moves to unphysical region and becomes IR unstable (bicritical):

\[ \epsilon = 4 - d \]

(IH, A modern approach to critical phenomena (CUP 2007), p. 53)

Can it be understood as an IR stable FP of another theory?
Fei, Giombi, Klebanov (PRD 2014): consider

\[ L = \frac{1}{2} (\partial_\mu z)^2 + \frac{1}{2} (\partial_\mu \phi_i)^2 + gz\phi_i\phi_i + \lambda z^3 \]

which is (log) renormalizable at $d=6$.

Below $d=6$ there is a IR stable fixed point for ($d = 6 - \epsilon$)

\[ N_{\text{crit}} = 1038.266 - 609.840 \epsilon - 364.173 \epsilon^2 + \mathcal{O}(\epsilon^3) \]

(Fei, Giombi, Klebanov, Tarnopolsky, PRD 2015)
Alternative formulation (IH and Janssen, PRD 2016)

Consider XY model (N=2):

\[
(\phi_1^2 + \phi_2^2)^2 = (\phi_1^2 - \phi_2^2)^2 + (2\phi_1\phi_2)^2 = (\phi^T \sigma_3 \phi)^2 + (\phi^T \sigma_1 \phi)^2.
\]

Alternative Hubbard-Stratonovich decoupling

\[-\frac{g^2}{2} (\phi_1^2 + \phi_2^2)^2 = \frac{1}{2} z_a z_a + g z_a \phi^T \sigma_a \phi \quad a \in \{1, 3\}\]

to motivate another representation of the XY model:

\[
L = \frac{1}{2} z_a (m_z^2 - \partial^2_{\mu}) z_a + \frac{1}{2} \phi_i (m_\phi^2 - \partial^2_{\mu}) \phi_i + g z_a \phi^T \sigma_a \phi
\]
For general $N$:

\[
\frac{1}{2} z_a z_a + g z_a \phi^T \Lambda^a \phi = -\frac{g^2}{2} \phi_i \Lambda^a_{ij} \phi_j \phi_k \Lambda^a_{kl} \phi_l
\]

\[a = 1, \ldots, M_N\]

\[M_N = (N - 1)(N + 2)/2\]

is the number of components of second rank irreducible tensor. Completeness of the set of real symmetric $\Lambda^a$ - matrices

\[\Lambda^a_{ij} \Lambda^a_{kl} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl}\]

So that

\[
\frac{1}{2} z_a z_a + g z_a \phi^T \Lambda^a \phi = g^2 \left(\frac{1}{N} - 1\right) (\phi_i \phi_i)^2
\]

is just the original quartic term!
Alternative $O(N)$ model:

\[ L = \frac{1}{2} (\partial_\mu z_a)^2 + \frac{1}{2} (\partial_\mu \phi_i)^2 + g z_a \phi_i \Lambda^a_{ij} \phi_j + \lambda \text{Tr}[(z_a \Lambda^a)^3]. \]

which is also log-renormalizable in $d=6$. Right below $d=6$, one loop:

\[ \frac{d\lambda}{d \ln b} = \frac{1}{2} (\epsilon - 3\eta_z) \lambda + 36 \left( N + 4 - \frac{24}{N} \right) \lambda^3 + \frac{4}{3} g^3, \]

\[ \frac{dg}{d \ln b} = \frac{1}{2} (\epsilon - \eta_z - 2\eta_\phi) g + 4 \left( 1 - \frac{2}{N} \right) g^3 + 12 \left( N + 2 - \frac{8}{N} \right) g^2 \lambda, \]

\[ \eta_z = 12 \left( N + 2 - \frac{8}{N} \right) \lambda^2 + \frac{4}{3} g^2, \quad \eta_\phi = \frac{4}{3} \left( N + 1 - \frac{2}{N} \right) g^2. \]
This flow has an IR stable FP for:

\[ 1 < N < 2.6534 \]

and again for

\[ 2.9991 < N < 3.6846 \]

For \( N=2 \):

\[ \eta_\phi = 2\eta_z = \frac{2}{5} \epsilon \]

and for \( N=3 \):

\[ \eta_z = \eta_\phi = \frac{5}{33} \epsilon \]

and positive!
Flow for $N=3$:

For $3.6847 < N < 4$ the fixed point $A$ becomes stable, but runs to infinity as $N \to 4$. 
Conclusion:

1) Two possible examples of fixed point collision and annihilation: a) interacting Luttinger fermions in semiconductors, and b) QED at low $N$; probably many other

2) Characteristic separation of scales; gaps are naturally small

3) Tensor representation of the $O(N)$ models: new road to IR fixed points above $d=4$?
Di Pietro et al PRL 2016: neglect of $e^4$ terms gives

1) Fixed points near $d=4$ are at the line $g_+ = 0$

2) Gaussian FP is pinned at $g_- = 0$

3) Critical point goes through it and destabilizes it at

$$1 - 3e_c^2 = 0$$

4) From the leading order beta function for the charge then

$$N_c = (9/4)(4 - d)$$
Yukawa-like field theory for the nematic (IR) critical point: (Janssen & IH, PRB 2015)

\[ L = L_\psi + L_{\psi\phi} + L_\phi \]

\[ L_\psi = \psi^\dagger (\partial_\tau + \gamma_\alpha d_\alpha (-i\nabla)) \psi, \]

\[ L_{\psi\phi} = g\phi_\alpha \psi^\dagger \gamma_\alpha \psi, \]

\[ L_\phi = \frac{1}{4} T_{ij} \left( -c\partial_\tau^2 - \nabla^2 + r \right) T_{ji} + \lambda T_{ij} T_{jk} T_{ki} \]

\[ + \mathcal{O}(T^4). \]

where the nematic tensorial order parameter is

\[ T_{ij} = \phi_\alpha \Lambda_{\alpha,ij} \quad \langle \phi_\alpha \rangle = \frac{-g}{r} \langle \psi^\dagger \gamma_\alpha \psi \rangle \]

And \( \Lambda_\alpha \) are the five three dimensional Gell-Mann matrices.
RG flow, close to four (spatial) dimensions:

“B”: “classical” nematic critical point \(\text{(Priest and Lubensky, 1976)}\)

“F”: new fermionic fixed point