Eigenstate Thermalization Hypothesis in Conformal Field Theory

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Universal Properties of “Chaotic” CFTs

Motivation

- build understanding of highly excited states in QFT

Plan of the talk

- what is Eigenstate Thermalization Hypothesis (ETH)?
- formal aspects of ETH in CFT
- phenomenological approach to eigenstate thermalization

Thermal State of Quantum Isolated System

Thermal isolated system is described by micro-canonical ensemble (all states from a narrow energy shell are equally probable)

$$\boxed{\rho^{\text{mic}} = N^{-1} \sum |E_n\rangle\langle E_n| \qquad E_n \in [E - \Delta E, E + \Delta E]}$$

- equivalence ensembles

$$\rho^{\text{mic}} \approx \rho^{\text{can}} = e^{-\beta H + F}$$

- all observables assume their thermal (micro)canonical values

$$\text{Tr}_{\bar{B}} \left(\rho_B A\right) \approx N^{-1} \sum_n \langle E_n | A | E_n \rangle = A^{\text{mic}}$$

Thermalization of Isolated System in Pure State

- unitary evolution $\Psi(t) = U(t)\Psi_0$ precludes emergence of ρ^{mic}

Eigenstate Thermalization Hypothesis

- individual energy eigenstates are thermal

$$\boxed{\langle E_n|A|E_m\rangle = A_{nm} = A^{\text{eth}}(E_n)\delta_{nm} + e^{-S/2}\, r_{nm}}$$

Deutsch'91 Srednicki'94; 99 Rigol, Dunjko, Olshanii'08

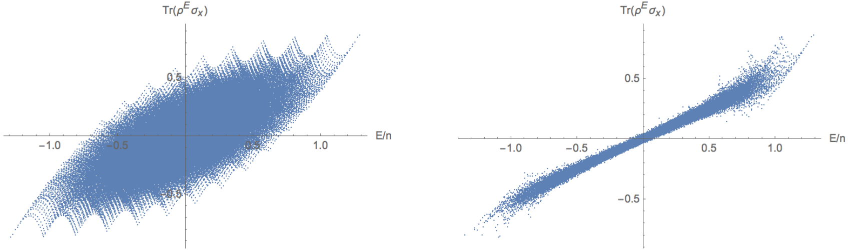
$$\langle\Psi|A(t)|\Psi\rangle = \sum_n |c_n|^2 A_{nn} + \sum_{n\neq m} e^{-i(E_n-E_m)t} c_n^* c_m A_{nm}$$

(micro)-canonical from the "diagonal" ensemble

$$\overline{\langle\Psi|A(t)|\Psi\rangle} = \sum |c_n|^2 A_{nn} \approx \left(\sum |c_n|^2\right) A^{\text{eth}} \approx A^{\text{mic}}$$

Integrability vs chaoticity (ETH)

diagonal matrix elements $A_{nn} = \langle E_n | A | E_n \rangle$ for integrable and non-integrable spin-chains



- emergence of ETH is correlated with Wigner level spacing (level repulsion)
- qualitatively, integrability = no correlation between neighboring levels, chaos = strong correlation

Subsystem ETH

- which operators A satisfy ETH?

few-body operators A are expected to satisfy ETH

exponentially many operators within a fixed subsystem satisfy ETH
Garrison-Grover'15

- universal form of the reduced density matrix

$$\boxed{||\rho^E - \rho^{\text{eth}}(E_n)|| = O(e^{-S/2})}$$

AD-Lashkari-Liu'16 $\rho^E = \text{Tr}_{\bar{B}}|E_n\rangle\langle E_n|$

-strong(est) formulation: list of operators follows

-volume-dependence of pre-factor

ETH in CFT

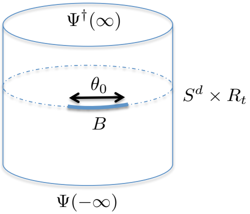
- Quantization of CFT on a cylinder $\mathbb{S}^d \times R_t$
- Operator-states correspondence

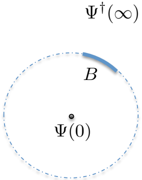
Local ETH

$$\langle \Psi | A | \Psi \rangle = \frac{\langle \Psi(\infty) A(1) \Psi(0) \rangle}{L^{\Delta} \langle \Psi(\infty) \Psi(0) \rangle} = \frac{C^{A}_{\Psi\Psi}}{L^{\Delta}}$$

$$C^{A}_{\Psi\Psi} = L^{\Delta} \left(A(E) + O(e^{-S/2}) \right)$$

"thermal" expectation value $A(E)$ is a smooth function of $E = h/L$





taking thermodynamic limit while keeping energy density fixed

- $\lambda^{-(d+1)} = E/L^d = h/L^{d+1}$, $\Delta = [A]$ – fixed
- $L \to \infty, \;\; E \to \infty, \;\; h = [\Psi] \to \infty$

Local ETH

$$\boxed{C^A_{\Psi\Psi} \approx h^{\Delta/(d+1)}\lambda^{\Delta} A(\lambda)}$$

- Ψ is a "heavy" scalar primary of dimension

$$h \sim (L/\lambda)^{d+1} \to \infty$$

- $C^A_{\Psi\Psi}$ can't grow with h faster then $h^{\Delta/(d+1)}$ lest there is no thermodynamic limit

Conjecture: "chaotic" CFTs satisfy local ETH

- for any primary A, $\langle\Psi|A|\Psi\rangle \approx A(\lambda)$ is a smooth function of thermal wave-scale λ (inverse of "effective temperature"), $A(\lambda) \sim \lambda^{-\Delta}$

Comments on Local ETH

- up to $1/L^d$ corrections $A(\lambda)$ is the thermal expectation value

$$\boxed{\langle\Psi|A|\Psi\rangle \approx \langle A\rangle_{T=\lambda^{-1}} = a_0\lambda^{-\Delta} \qquad C^A_{\Psi\Psi} = a_0 h^{\Delta/(d+1)}}$$

- in $2d$ thermal expectation values of (quasi)-primaries (outside of identity block) is zero
- free theories do not satisfy ETH
- descendant states $\Psi = \partial^\ell \Psi_0$ satisfy ETH (are thermal) only with the precision $O(\ell/h_0)$

- Insight for/from conformal bootstrap?

- Probing CFT ETH with lattice models?

Relation between subsystem ETH and local ETH

- assuming stronger version, subsystem ETH,
  $||\rho^\Psi - \rho(\lambda)|| = O(e^{-S/2}), \quad \rho^\Psi = \mathrm{Tr}_{\bar{B}}|\Psi\rangle\langle\Psi|,$
  one can prove local ETH

$$\mathrm{Tr}_{\bar{B}}\left((\rho^\Psi - \rho(\lambda))A\right) = O(e^{-S/2})$$

Proof:

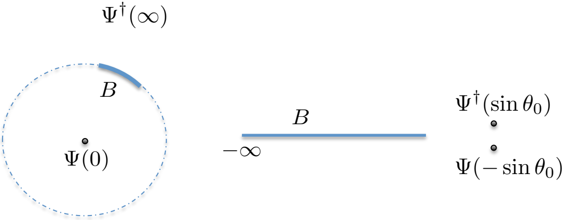
$$\mathrm{Tr}_{\bar{B}}\left((\rho^\Psi - \rho(\lambda))A\right) \leq \left||\rho^\Psi - \rho(\lambda)\right||^{1/2}\left(\mathrm{Tr}_{\bar{B}}\left((\rho^\Psi + \rho(\lambda))A^2\right)\right)^{1/2}$$

- assuming weaker version, local ETH, can one prove subsystem ETH?

One can use OPE to reduce $\langle\Psi|A|\Psi\rangle$ for composite A to expectation values of local operators, but strictly speaking this is not enough

Density Matrix as an Euclidean Path Integral

- reduced density matrix ρ^Ψ is an Euclidean path integral over flat space with two insertions



- conformal transformation to Rindler frame $\omega = \frac{z-q}{qz-1}$, $q = e^{i\theta_0}$, $\theta_0 = l/L$, brings "heavy" operators together

$$\rho^\Psi = \frac{\Psi(q)\Psi(\bar{q})}{\langle\Psi(q)\Psi(\bar{q})\rangle} = 1 + \sum_k (2\sin\theta_0)^{\Delta_k} C^k_{\Psi\Psi} O_k(\bar{q}) \approx \rho(\lambda)$$

Nima Lashkari, PRL 113, 051602 (2014), AD-Lashkari-Liu'16

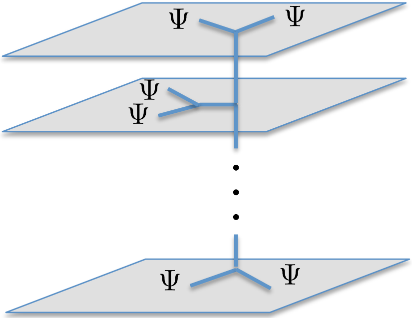
Proximity of Density Matrices

- Pinsker inequality bounds the difference between density matrices in terms of relative entropy

$$\left|\left|\rho^{\Psi} - \rho(\lambda)\right|\right|^2 \leq 2S\left(\rho^{\Psi}||\,\rho(\lambda)\right)$$

- relative Renyi entropy may be represented as an Euclidean path integral over sphere with n insertions
  Nima Lashkari, PRL 117, 041601 (2016)

- exponential precision of OPE expansion results in exponential suppression of $S\left(\rho^{\Psi}||\,\rho(\lambda)\right)$ in all d
  AD-Lashkari-Liu'16



Universality of $\rho(\lambda)$ in $2d$

- OPE expansion of $\Psi(q)\Psi(\bar{q})$ simplifies significantly in the thermodynamic limit $\theta_0 = l/L$, $C^k_{\Psi\Psi} \sim (L/\lambda)^{\Delta_k}$, $L \to \infty$

$$\rho^\Psi = 1 + \sum_k (2\sin\theta_0)^{\Delta_k} C^k_{\Psi\Psi} O_k(\bar{q}) \approx \rho(\lambda)$$

- at leading order only (quasi)-primary operators contribute
- in $2d$, *assuming ETH*, only Virasoro descendants of identity contribute (Fitzpatrick-Kaplan-Walters'15)

$$\rho^\Psi \approx \rho(\lambda) = 1 + \sum_k (2l/\sqrt{2\pi}\lambda)^{2k} \frac{c^k}{d_{2k}} T^k(\bar{q})$$

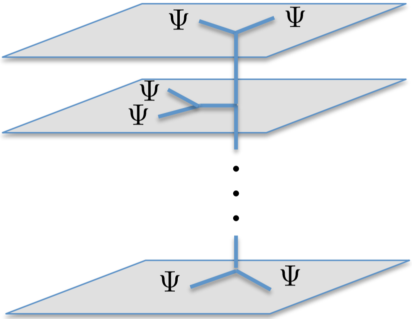
T^k are (quasi)-primaries that include $:T^k:$ for example
$T^2 = (TT) - 3\partial^2 T/10$

Renyi entropy for $\rho(\lambda)$ in $2d$

- universal density matrix $\rho(\lambda)$ can be used to calculate vacuum subtracted Renyi entropies in the excited state Ψ
- n-th Renyi entropy – Euclidean integral over a sphere $\tilde{\omega} = \omega^{1/n}$ with $2n$ insertions
- the result can be expanded in l/λ

$$(1-n)\Delta S_n =$$

$$\log\left[n^{-4nh}\frac{\langle\prod_i \Psi(q^{i/n})\Psi(\bar{q}^{i/n})\rangle}{\langle\Psi(q)\Psi(\bar{q})\rangle^n}\right]$$



Comparing Renyis for $\rho(\lambda)$ and ρ^T in $2d$

- Renyi entropy for $\rho(\lambda)$, expansion in $x = (2l/\lambda)$

$$\boxed{\Delta S_n = \frac{(1+n)c}{12\pi n}x^2 - \frac{(1+n)c}{120\pi^2 n}x^4 \cdot \frac{n^2+11}{12n^2} + \frac{(1+n)c}{630\pi^3 n}x^6 \cdot \frac{(4-n^2)(n^2+47)}{144n^2} + \ldots}$$

- Renyi entropy for ρ^T

$\Delta S_n = \frac{(1+n)c}{6n}\log\left[\sinh(2\pi l/\beta)/(2\pi l/\beta)\right]$, matching β and λ yields

$$\boxed{\Delta S_n = \frac{(1+n)c}{12\pi n}x^2 - \frac{(1+n)c}{120\pi^2 n}x^4 + \frac{(1+n)c}{630\pi^3 n}x^6 + \ldots}$$

- Renyi entropy for $n \neq 1$ do not coincide
- Entanglement entropy $n = 1$ is the same
- Pinsker inequality $\left|\left|\rho(\lambda) - \rho^T\right|\right|^2 \leq \Delta H - \Delta S \to 0$

$$\Delta H = \langle\Psi|H|\Psi\rangle - \mathrm{Tr}(\rho^T H) \qquad H = \beta\int_{-\infty}^{0} dx(1 - e^{2\pi x/\beta})T_{00}$$

Upshot: ETH in $2d$

- at leading order in $1/L$, universal density matrix $\rho(\lambda)$ includes only quasi-primaries T^k
- universal density matrix $\rho(\lambda)$ is trace-distance close to thermal density matrix ρ^T

this ensures all observables in heavy state Ψ confined to a given sub-system are thermal

$$\text{ETH in } 2d \text{ CFT: } C^A_{\Psi\Psi} \sim h^{\Delta/2}0 + \dots$$

mismatch of higher Renyi entropies is not a contradiction

FannesAudenaert inequality:

$$\left|S(\rho(\lambda)) - S(\rho^T)\right| \leq \left|\left|\rho(\lambda) - \rho^T\right|\right| \log \dim \mathcal{H}_B$$

$$\left|S_n(\rho(\lambda)) - S_n(\rho^T)\right| \leq \left|\left|\rho(\lambda) - \rho^T\right|\right| (\dim \mathcal{H}_B)^{1-1/n}$$

ETH in $d > 2$

- at leading order, only primaries contribute to $\rho(\lambda)$
- thermal expectation values of primaries may not be zero

"Phenomenological approach": leaving only "products" of $T_{\mu\nu}$ in $\rho(\lambda)$

$$\begin{aligned} \rho^{\Psi} &= 1 + \sum_k (2\sin\theta_0)^{\Delta_k} C^k_{\Psi\Psi} O_k(\bar{q}) \approx \\ \rho(\lambda) &= 1 + \frac{d+1}{d}\left(\frac{2l}{\lambda}\right)^{d+1} T_{\mu\nu} n^{\mu} n^{\nu} + \ldots \end{aligned}$$

$\rho(\lambda)$ can be systematically defined in "holographic" large N theories

Entanglement entropy in $d > 2$

- Entanglement entropy for a holographic black hole background, expanded in powers of l
  Blanco-Casini-Hung-Myers'13

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + dx^2 + f^{-1}(x)dz^2\right), \quad f = 1+(z/z_0)^{d+1}$$

- Entanglement entropy for $\rho(\lambda)$, expanded in l/λ

$$\boxed{\Delta S = \frac{4\pi^{d/2+1}C}{\Gamma(d/2)d(d+2)}\left(\frac{l}{\lambda}\right)^{d+1} - \frac{(d+1)^2\Gamma(d+3)\Gamma(d)C}{\Gamma(5+2d)}\left(\frac{l}{\lambda}\right)^{2(d+1)} + \dots}$$

Holographic result matches exactly EE $S[\rho(\lambda)]$ for the universal $\rho(\lambda)$

Conclusions

- equivalence between different definitions of ETH for CFTs
  - examples of "chaotic" theories
  - connection to bootstrap?
- heavy states Ψ behave as thermal when $\Psi\Psi$ OPE includes higher "powers" of $T_{\mu\nu}$
  - phenomenological (uncontrolled) approximation to study thermalization dynamics in CFT
  - new angle to study heavy states in the context of AdS/CFT
- universal form of $\rho(\lambda)$ in $2d$ and $d > 2$
  - $\rho(\lambda)$ is well-defined in $2d$ for finite and infinite c
  - universality of thermal state $\rho(\lambda)$ at large central charge in $d > 2$?