Chaos and the CFT Bootstrap

A 2D Perspective on the SYK model

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20 Years Later: The Many Faces of AdS/CFT
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Based on: arXiv:1705.08408, T. Mertens, J. Turiaci, HV + work in progress with H-T Lam
We study the geometric quantization of Teichmüller space and show that the physical state conditions take the form of conformal Ward identities that define the space of Virasoro conformal blocks in 2-d CFT. Possible applications of these results to the [conformal bootstrap] are indicated.

Hilbert state of the (2 + 1)-dimensional gravity theory

$$\Psi \in \mathcal{H}^+ \otimes \mathcal{H}^-$$  \hspace{1cm} (6.13)

can be decomposed into a sum of left and right conformal blocks as

$$\Psi = \sum_{I,J} N^{IJ} \Psi_I^+ \otimes \Psi_J^-,$$  \hspace{1cm} (6.14)
\[ \left\langle \mathcal{O}_1(0) \mathcal{O}_2(1) \mathcal{O}_3(z, \bar{z}) \mathcal{O}_4(\infty) \right\rangle = \sum_a \left| \begin{array}{c} 2 \\ a \\ 1 \\ b \\ 4 \end{array} \right| ^2 \]

Conformal blocks

\[ \sum_b F_{ab} \left[ \begin{array}{c} 2 \\ 3 \\ 1 \\ 4 \end{array} \right] \]

\[ F = \text{Fusion matrix} \quad \text{R} = \text{Braid matrix} \]

\[ \sum_b R_{ab} \left[ \begin{array}{c} 2 \\ 3 \\ 1 \\ 4 \end{array} \right] \]
2D Virasoro CFT = 2D Quantum Hyperbolic Geometry

\[ T(z) = \sum_{i=1}^{n-1} \left( \frac{\Delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right) \]

Stress-energy tensor

Elliptic

Hyperbolic
2+1-D AdS Gravity = 2D Quantum Hyperbolic Geometry

\[ \hat{l}_\alpha |\alpha\rangle = l_\alpha |\alpha\rangle. \]

\[ \hat{l}_\beta |\beta\rangle = l_\beta |\beta\rangle. \]

\[
\mathcal{R}_{\alpha\beta} = \exp\left(\frac{i}{\hbar} S_{\alpha\beta}(l_\alpha, l_\beta)\right) = \langle \beta | \alpha \rangle
\]

\[
S_{\alpha\beta} = \text{Vol}\left(T\left[\begin{array}{ccc} 1 & 2 & \alpha \\ 3 & 4 & \beta \end{array}\right]\right)
\]

Volume of a hyperbolic tetrahedron

Ponsot-Teschner

6j-symbol of SL(2)_q
Exchange relation for localized wave-packets

- contains the gravitational scattering amplitude
- spectral decomposition of OTO four-point function
- scattering phase determined via geometric optics

\[
\phi_{\omega-\alpha}(t_1) \phi_{\alpha}(t_0) = e^{\frac{i}{\hbar} S_{\alpha\beta}} \phi_{\omega-\beta}(\tilde{t}_0) \phi_{\beta}(\tilde{t}_1).
\]
String Theory in AdS$_3$

Gravity dominated regime near BH horizon
Long range properties

2+1 Quantum Gravity

Holographic Dictionary

Discrete Spectrum

Irrational 2D CFT

Entropy dominated regime $c>>1$, $\Delta > c/12$
Modular properties

Discrete Spectrum

Virasoro Liouville CFT

Continuous Spectrum

Modular Bootstrap

S. Jackson, L. McGough, HV NPB 901 (2015) 382;
J. Turiaci, HV, JHEP 1612 (2016) 110
2D Quantum Dilaton Gravity

Gravity dominated regime, effective geometric description

Holographic Dictionary

Discrete Spectrum

Continuum Spectrum

SYK model

Effective IR dynamics dominated by Goldstone mode

Schwarzian Quantum Mechanics
SYK model = 1D many body QM with maximal chaos

\[ H = \sum_{ijkl} J_{ijkl} \psi^i \psi^j \psi^k \psi^\ell \]

\( \{ \psi^i, \psi^j \} = \delta^{ij} \)

random couplings

N majorana variables

Large N limit of SD equations = soluble

Dominated by `pumpinic` diagrams
IR limit of SD equations

\[ \int d\tau' G(\tau, \tau') \Sigma(\tau', \tau'') = -\delta(\tau - \tau'') \, , \quad \Sigma(\tau, \tau') = J^2 [G(\tau, \tau')]^{q-1} \]

are invariant under 1D diffeomorphisms

\[ G(\tau, \tau') \rightarrow [f'(\tau)f'(\tau')]^\Delta G(f(\tau), f(\tau')) \, , \quad \Sigma(\tau, \tau') \rightarrow [f'(\tau)f'(\tau')]^{\Delta(q-1)} \Sigma(f(\tau), f(\tau')) \]

→ IR effective theory is dominated by a dynamical Goldstone mode  = 1D reparametrizations \( f(\tau) \)
The Schwarzian theory is integrable and expected to be exactly soluble at any value of the energy spectrum and dynamics are thus uniquely determined by the section 2, the Hamiltonian of di...
1 Introduction: Schwarzian QM

\[
L = \pi_\phi \dot{\phi} + \pi_f \dot{f} - (\pi_\phi^2 + \pi_f e^\phi)
\]

SL(2,R) symmetry: \( f \rightarrow \frac{af + b}{cf + d} \) \( \rightarrow \) generators \( [\ell_a, \ell_b] = i \epsilon_{abc} \ell_c \)

Hamiltonian = Casimir:

\[
H = \pi_\phi^2 + \pi_f e^\phi = \ell_0^2 - \frac{1}{2} \{\ell_{-1}, \ell_1\}
\]

\[
j = -\frac{1}{2} + ik \quad \quad E(k) = -j(j + 1) = \frac{1}{4} + k^2
\]
\[ Z(\beta) = \int_{\mathcal{M}} \mathcal{D}f \, e^{-S[f]} \]

Partition function

\[ \mathcal{M} = \text{Diff}(S^1)/\text{SL}(2, \mathbb{R}) \]

integral over energy \( E = \frac{1}{4} + k^2 \)

with continuous spectral density

\[ \rho(E) = \sinh(2\pi \sqrt{E - 1/4}) \]

Stanford, Witten

\[ Z(\beta) = \int_{0}^{\infty} d\mu(k) \, e^{-\beta E(k)}, \quad d\mu(k) = dk^2 \sinh(2\pi k). \]
The spectrum of states in the Schwarzian theory arises from the CFT spectrum of states with conformal dimension \(\Delta = \frac{c-1}{24} + b(\Delta E)\), in the limit \(b \to 0\). The operators in the Schwarzian are all light CFT operators with conformal dimension \(\Delta \approx b\).

The second formula has a clear physical significance. The large \(c\) limit sends \(\tilde{q} \to 0\), which turns the operator \(\tilde{q}_L\) into a projection operator on the lowest energy state in the given channel. Combining (3.11), (3.16) and (3.19) we obtain that

\[
Z(\Delta) = Z_1 \int d\mu(k) e^{\Delta E(k)},
\]

\[
d\mu(k) = dk_2 \sinh(2\pi k),
\]

reproducing the result obtained in [23].

While the explicit formula (3.20) for the spectral density is not a new result, our derivation provides a new and useful perspective on the Schwarzian theory. Specifically, it indicates that the 1D model arises as a special \(c \to 1\) limit of 2D Virasoro CFT, in which we only keep the states with conformal dimensions close to the threshold \(c\).

The above modular bootstrap argument identifies a natural spectral density on the space of Virasoro representations, given by the modular S-matrix element \(\mathcal{S}_{P|0}\) [32]. This spectral density is not a specific property of a particular 2D CFT, but a universal measure analogous to the Plancherel measure on the space of continuous series representations of \(\text{SL}(2,\mathbb{R})\). This measure is defined for any value of the central charge \(c\). We have shown that, after taking the large \(c\) limit while zooming in close to \(c = \frac{c-1}{24}\), it coincides with the exact spectral density of the Schwarzian theory. In the following sections we will generalize this observation with the aim of studying correlation functions.
Partition function = integral over a symplectic manifold $\mathcal{M}$ can be quantized!

$$Z(\beta) = \int_{\mathcal{M}} \mathcal{D} f \ e^{-S[f]}$$

Identity representation

$$= \lim_{c \to \infty, q \to 1} \text{Tr}(q^{L_0}),$$

$$q^{\frac{c}{24}} = e^{-\frac{\pi^2}{\beta}} = \text{fixed}.$$
\[
\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{1}{Z} \int_{\mathcal{M}} Df \ e^{-S[f]} \mathcal{O}_1 \ldots \mathcal{O}_n
\]

**Correlation functions**

\[
\mathcal{O}(\tau_1, \tau_2) \equiv \left( \frac{\sqrt{f'(\tau_1)f'(\tau_2)}}{\frac{\beta}{\pi} \sin \frac{\pi}{\beta} [f(\tau_1) - f(\tau_2)]} \right)^{2\ell}
\]

**Two-point function**

\[
\langle \mathcal{O}_\ell(\tau_1, \tau_2) \rangle = \int \prod_{i=1}^{2} d\mu(k_i) \ A_2(k_i, \ell, \tau_i).
\]
Liouville theory on hyperbolic cylinder \(\rightarrow\) reduces dilaton gravity for \(c \to \infty\)

\[
S = \frac{c}{192\pi} \int d\tau \int_0^\pi d\sigma \left[ (\partial \phi)^2 + 4\mu e^{2\phi} \right]
\]

\[
\partial_u \partial_v \phi(u, v) = e^{2\phi(u, v)}.
\]

\[
e^{\phi_{cl}(u,v)} = \frac{\sqrt{f'(u)f'(v)}}{\beta} \sin \frac{\pi}{\beta} [f(u) - f(v)].
\]

Insertion of \(O_\ell(\tau_1, \tau_2)\) in Schwarzian \(\leftrightarrow\) Insertion of \(V_\ell = e^{2\ell \phi(\tau_1, \tau_2)}\) in Liouville CFT
Two point function

\[
\langle \mathcal{O}_\ell(\tau_1, \tau_2) \rangle = \int \prod_{i=1}^{2} d\mu(k_i) \ A_2(k_i, \ell, \tau_i).
\]

\[
A_2(k_i, \ell, \tau_i) = e^{-(\tau_2 - \tau_1)k_1^2 - (\beta - \tau_2 + \tau_1)k_2^2} \frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}.
\]
The exact non-perturbative answer for the 2n-point functions can be summarized via a simple set of diagrammatic rules:

\[ k \sim e^{-k^2(\tau_2 - \tau_1)} \]

`propagator`

\[ \ell \quad \gamma_{\ell}(k_1, k_2) \]

`vertex`

\[ \gamma_{\ell}(k_1, k_2) = \sqrt{\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)}}. \]
Four-point function

\[ \langle \mathcal{O}_{\ell_1}(\tau_1, \tau_2) \mathcal{O}_{\ell_2}(\tau_3, \tau_4) \rangle = \]

OTO four-point function

\[ \langle \mathcal{O}_{\ell_1}(\tau_1, \tau_2) \mathcal{O}_{\ell_2}(\tau_3, \tau_4) \rangle_{\text{OTO}} = \]
The procedure can be graphically represented as the integral representation of the out-of-time-ordered four-point function. The total calculation that the conformal block inside the integral in (5.6) becomes trivial for the full conformal block, we can use the R-matrix transformation of Ponsot and Teschner. The cross ratio becomes infinite. Even though we do not know the explicit expression for the time-ordered case. Inserting the two operators in opposite order gives the 2D conformal block. The argument of the s-channel conformal block is continued to an OTO conformal block. Here the OTO label indicates that we have applied a specific monodromy transformation of this monodromy transformation in the Schwarzian view of the 2D CFT, this means that one of the chiral conformal blocks has been analytically continued to an OTO conformal block.

The four-point function in the Schwarzian theory corresponds to a two-point function. As before, we can go to the closed string channel and write the four point function as

\[
G^\text{OTO}_{\ell_1 \ell_2} = \int dPdQ \, \Psi^\dagger_{ZZ}(P)\Psi_{ZZ}(Q) \times \int dP_s \, R_{P_s P_t}
\]

\[
= \int dPdQ \, \Psi^\dagger_{ZZ}(P)\Psi_{ZZ}(Q) \times \int dP_s dP_t \, R_{P_s P_t}
\]

where

\[
\mathcal{F}^\ell(P_{\ell_1}, P_{\ell_2}, P_{\ell_3}, P_{\ell_4}) = \int dPdQ \, \Psi^\dagger_{ZZ}(P)\Psi_{ZZ}(Q) \times \int dP_s dP_t \, R_{P_s P_t}
\]
R-matrix

\[
R_{k_s k_t} \left[ \frac{k_4}{k_1}, \frac{\ell_2}{\ell_1} \right] = \left\{ \frac{\ell_1}{\ell_2}, \frac{k_4}{k_1}, \frac{k_s}{k_t} \right\} = \sqrt{\Gamma(\ell_1 \pm ik_2 \pm ik_s) \Gamma(\ell_3 \pm ik_4 \pm ik_t) \Gamma(\ell_1 \pm ik_4 \pm ik_t) \Gamma(\ell_3 \pm ik_4 \pm ik_s)} \times \mathbb{W}(k_s, k_t; \ell_1 + ik_4, \ell_1 - ik_4, \ell_3 - ik_2, \ell_3 + ik_2),
\]

\[\mathbb{W} = \text{Wilson function}
\]

linear combination of \( _4F_3 \)

\text{Matches with the gravitational shockwave amplitude}

Groenevelt

The R-matrix of the Schwarzian is found to be equal to a classical 6j-symbol of SU(1,1)
Semiclassical limit of OTO 4pt function

\[
\langle V_1 W_3 V_2 W_4 \rangle = \int_0^\infty dq_+ \int_0^\infty dp_- \Psi_1^*(q_+) \Phi_3^*(p_-) S(p_-, q_+) \Psi_2(q_+) \Phi_4(p_-)
\]

\[S = \exp \left( \frac{i\beta}{4\pi C} p_- q_+ \right)\]

Dray-'t Hooft S-matrix

\[C \sim G_N^{-1} \to \infty\]

[Shenker, Stanford]
Semiclassical limit of OTO 4pt function

\[ \langle V_1 W_3 V_2 W_4 \rangle = \prod_{i=1}^{4} \int \frac{d\omega_i}{2\pi} \Psi_1^*(\omega_1) \Psi_3^*(\omega_3) \, S(\omega_1, \omega_2, \omega_3, \omega_4) \, \Psi_2(\omega_2) \, \Psi_4(\omega_4). \]

Schwarchild S-matrix

\[ S(\omega_1, \omega_3; \omega_2, \omega_4) = \frac{\beta}{(2\pi)^2} \delta(\omega_1 + \omega_3 - \omega_2 - \omega_4) \frac{\Gamma(i\nu_1 - i\nu_2)}{(4\pi i C)^{i(-\nu_1 + \nu_2)}}. \]

Large C high temperature

\[ S = \exp \left( \frac{i\beta}{4\pi C} p_{-q_+} \right) \]
• High-temperature limit of the Schwarzian = 2D Virasoro CFT on the strip

\[
\frac{\langle V_1 V_2 W_3 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = z^{2\Delta_V} \mathcal{F}\left(\frac{V}{W}, \text{id}, x\right), \quad x = -\frac{\sinh \frac{\pi t_{12}}{\beta}}{\sinh \frac{\pi t_{32}}{\beta}} \frac{\sinh \frac{\pi t_{34}}{\beta}}{\sinh \frac{\pi t_{41}}{\beta}}
\]

Same semiclassical limit as the shockwave

[Maldacena, Stanford, Yang; Chen, Fitzpatrick, Kaplan, Li, Wang 16]
Microscopic understanding of Lyapunov and fast thermalizing behavior?

Figure 4: The scrambling of a signal (operator $A$) due to the a perturbation (operator $B$) at some earlier time $t_1 < t_0$. An observer that measures the state can detect signal $A$ only if $A$ acts on the state from the left. Passing $A$ through $B$ produces a new intermediate channel with energy $\beta$, which for $t_0 - t_1 > t_{\text{crit}}$ exceeds $\omega$. Signal $A$ becomes scrambled: its coherent phase information get washed out by the large entropy region of the spectrum near $M + \beta$. 

\[ B_{\omega-\alpha} A_\alpha |M\rangle \rightarrow A_{\omega-\beta} B_\beta |M\rangle \]

\[ B_{\omega-\beta} A_\alpha \]

\[ A_{\omega-\beta} B_\beta \]

\[ M + \beta \]

\[ M + \omega \]

\[ M + \alpha \]

\[ M \]