Early Results on Correlation Functions

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20 Years Later: The Many Faces of AdS/CFT

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OUTLINE:

A. 3-point correlators from
"Correlation Functions in the AdS/CFT Correspondence,”
DZF, Mathur, Matusis, Rastelli, 9804058.

B. 2-point correlators from idem.

C. Simplest example of exchange diagrams for 4-point functions
from "How to succeed at z-integrals without really trying,”
D’Hoker, DZF, Rastelli, 9905049.
Basic Facts of AdS/CFT:

1. Euclidean $AdS_{d+1}$ metric: 
   \[ ds^2 = \left( \frac{L^2}{z^2_0} \right) [dz_0^2 + \sum_{i=1}^{d} dz_i^2] \]
   Boundary at $z_0 = 0$.

2. Invariant under $SO(d+1,1)$ and discrete inversion (!):
   \[ z'_\mu = \frac{z_\mu}{z^2} \quad z'^2 = \frac{1}{z^2} \quad z^2 = z_\mu \delta^{\mu\nu} z_\nu \]

Jacobian:

\[ \frac{\partial z'_\mu}{\partial z_\nu} = z'^2 l_{\mu\nu}(z') \quad l_{\mu\nu}(z) = \delta_{\mu\nu} - \frac{2 z_\mu z_\nu}{z^2} = l_{\mu\nu}(z') \]

\[ l_{\mu\nu} = l_{\nu\mu} \quad l_{\mu\rho} l_{\rho\nu} = \delta_{\mu\nu} \quad \text{det} \; l_{\mu\nu} = -1 \]

\( \implies \) Inversion $\in O(d+1,1)$, but not $SO(d+1,1)$.
3. CFT operators transform under inversion as

\[ x_i \rightarrow x'_i = \frac{x_i}{x^2} \quad  O_\Delta(\vec{x}) \rightarrow O'_\Delta(\vec{x}) = |\vec{x}'|^{2\Delta} O_\Delta(\vec{x}') \]

Correlators behave as

\[ \langle O_{\Delta_1}(\vec{x}_1) \ldots O_{\Delta_n}(\vec{x}_n) \rangle = |\vec{x}'|^{2\Delta_1} \ldots |\vec{x}'|^{2\Delta_n} \langle O_{\Delta_1}(\vec{x}'_1) \ldots O_{\Delta_n}(\vec{x}'_n) \rangle \]

Translation and inversion symmetry determine

\[ \langle O_\Delta(\vec{x})O_\Delta(\vec{y}) \rangle = \frac{c_2}{(\vec{x} - \vec{y})^{2\Delta}} \]

\[ \langle O_{\Delta_1}(\vec{x})O_{\Delta_2}(\vec{y})O_{\Delta_3}(\vec{z}) \rangle = \frac{c_3}{|\vec{x} - \vec{y}|^{\Delta_{12}}|\vec{y} - \vec{z}|^{\Delta_{23}}|\vec{z} - \vec{x}|^{\Delta_{31}}} \]

with \( \Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3 \), etc.
4. CFT correlators are obtained from calculations in the AdS bulk by the following procedure:

i. The bulk dual of an operator $\mathcal{O}_\Delta$ is a scalar field $\phi_\Delta(z)$ in AdS. Its mass and scale dimension are related by $m^2 = \Delta(\Delta - d)$. Under inversion, $\phi_\Delta(z) \rightarrow \phi'_\Delta(z') = \phi_\Delta(z)$.

ii. Bulk dynamics is governed by action

$$S[\phi] = \frac{1}{2} \int \frac{d^{d+1}z}{z_0^{d+1}} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \frac{1}{3} \phi^3 + \ldots].$$

One must find solution of classical EOMs which satisfies a modified Dirichlet problem with boundary data $\varphi_\Delta(\vec{z})$:

$$\phi_\Delta(z_0, \vec{z}) \rightarrow z_0^{d-\Delta} \varphi_\Delta(\vec{z}) \text{ as } z_0 \rightarrow 0.$$

The bdy data $\varphi_\Delta(\vec{z})$ is source of dual operator $\mathcal{O}_\Delta(\vec{z})$.

iii. Bulk action, evaluated on-shell, is the generating functional of CFT correlators

$$\langle \mathcal{O}_\Delta(\vec{x}_1) \ldots \mathcal{O}_\Delta(\vec{x}_n) \rangle = (-1)^{n-1} \frac{\delta}{\delta \varphi_\Delta(\vec{x}_1)} \ldots \frac{\delta}{\delta \varphi_\Delta(\vec{x}_n)} S[\phi_\Delta] \varphi_\Delta = 0.$$
Perturbative AdS/CFT computations require bulk-bdy. + bulk-bulk propagators and Witten diagrams.

Bulk-bdy. prop. is a solution of \((\square - m^2)K_\Delta(z, \vec{x}) = 0\) with bdy. behavior:

\[ K_\Delta(z_0, \vec{z}; \vec{x}) \to z_0^{d-\Delta} \delta(\vec{z} - \vec{x}). \]

It is given by (Witten)

\[ K_\Delta(z_0, \vec{z}; \vec{x}) = c_\Delta \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \quad \text{with} \quad c_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \]

Using \(K_\Delta\) one constructs free-field solution of bdy value problem

\[ \phi_\Delta(z) = \int d^d\vec{x} K_\Delta(z_0, \vec{z}; \vec{x}) \varphi_\Delta(\vec{x}). \]

Under Inversion, \(z_\mu = z'_\mu/z'^2\), \(x_i = x'_i/x^2\), \(K_\Delta(z, \vec{x})\) transforms as a scalar field at \(z\) and as an operator of scale dim. \(\Delta\) at \(\vec{x}\):

\[ K_\Delta(z, \vec{x}) = |\vec{x}'|^{2\Delta} K_\Delta(z', \vec{x}'). \]
A. 2-point correlators require a special procedure (c.f. later in this talk), so we first consider 3-point correlators described by Witten diagram.

Diagram corresponds to integral (over region $w_0 > 0$):

$$A(\vec{x}, \vec{y}, \vec{z}) = \int \frac{dw_0 d^d w}{w_0^{d+1}} (\frac{w_0}{w_0^2 + (\vec{w} - \vec{x})^2})^{\Delta_1} (\frac{w_0}{w_0^2 + (\vec{w} - \vec{y})^2})^{\Delta_2} (\frac{w_0}{w_0^2 + (\vec{w} - \vec{z})^2})^{\Delta_3}$$

Two questions:

i. Does $A(\vec{x}, \vec{y}, \vec{z})$ transform properly under conformal trfs?

ii. Can one evaluate the integral and thus determine the dynamical constant $c_3$ in

$$\langle \mathcal{O}_{\Delta_1}(\vec{x})\mathcal{O}_{\Delta_2}(\vec{y})\mathcal{O}_{\Delta}(\vec{z}) \rangle = \frac{c_3}{|\vec{x} - \vec{y}|^{\Delta_12}|\vec{y} - \vec{z}|^{\Delta_23}|\vec{z} - \vec{x}|^{\Delta_31}}?$$
Two answers:
i. Use previous \( K_\Delta(z, \vec{x}) = |\vec{x}'|^{2\Delta} K_\Delta(z', \vec{x}') \) and invariance of measure \( \frac{d^{d+1}w}{w_0^{d+1}} = \frac{d^{d+1}w'}{w_0'^{d+1}} \) to obtain immediately

\[
A(\vec{x}, \vec{y}, \vec{z}) = |\vec{x}'|^{\Delta_1} |\vec{y}'|^{\Delta_2} |\vec{z}'|^{\Delta_3} A(\vec{x}', \vec{y}', \vec{z}').
\]

ii. Integral with 3 denominators and restriction to \( w_0 > 0 \) is difficult. But one can simplify by using translation invariance to move \( \vec{z} \rightarrow \vec{0} \):

\[
A(\vec{x}, \vec{y}, \vec{z}) = A(\vec{x} - \vec{z}, \vec{y} - \vec{z}, \vec{0}) \equiv A(\vec{u}, \vec{v}, \vec{0})
\]

The \( \Delta_3 \) factor simplifies to

\[
\left( \frac{w_0}{w_0^2 + (\vec{w} - \vec{z})^2} \right)^{\Delta_3} \rightarrow \left( \frac{w_0}{\vec{w}^2} \right)^{\Delta_3} = (w_0')^{\Delta_3}. \quad \leftarrow \text{no denominator!}
\]

Translated integral after inversion of \( w_\mu, \vec{u}, \vec{v} \):

\[
A(\vec{u}, \vec{v}, \vec{0}) = \frac{1}{|\vec{u}|^{2\Delta_1} |\vec{v}|^{2\Delta_2}} \int \frac{d^{d+1}w'}{w_0'^{d+1}} \left( \frac{w_0'}{w_0'^2 + (\vec{w}' - \vec{u}')^2} \right)^{\Delta_1} \left( \frac{w_0'}{w_0'^2 + (\vec{w}' - \vec{v}')^2} \right)^{\Delta_2} w_0'^{\Delta_3}
\]
\[ A(\vec{u}, \vec{v}, \vec{0}) = \frac{1}{|\vec{u}|^{2\Delta_1} |\vec{v}|^{2\Delta_2}} \int \frac{d^{d+1}w'}{w'_0} \left( \frac{w'_0}{w'_0^2 + (\vec{w}' - \vec{u}')^2} \right)^{\Delta_1} \left( \frac{w'_0}{w'_0^2 + (\vec{w}' - \vec{v}')^2} \right)^{\Delta_2} w'_0 \Delta_3 \]

Two denoms. easily handled with Feynman parameters to give:

\[ A(\vec{u}, \vec{v}, \vec{0}) = \frac{a(\Delta_1, \Delta_2, \Delta_3)}{|\vec{u}|^{2\Delta_1} |\vec{v}|^{2\Delta_2} |\vec{u}' - \vec{v}'|^{\Delta_1 + \Delta_2 - \Delta_3}} = \frac{a(\Delta_1, \Delta_2, \Delta_3)}{|\vec{x} - \vec{y}|^{\Delta_{12}} |\vec{y} - \vec{z}|^{\Delta_{23}} |\vec{z} - \vec{x}|^{\Delta_{31}}} \]

with

\[ a(\Delta_1, \Delta_2, \Delta_3) = \frac{\pi^{d/2}}{2} \frac{\Gamma(\Delta_1 + \Delta_2 + \Delta_3 - d/2) \Gamma(\Delta_{12}/2) \Gamma(\Delta_{23}/2) \Gamma(\Delta_{31}/2)}{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3)} \]

Not very pretty, but it contains physical information, as we will see.
B. Standard procedure for 2-pt. functions (due to GKP), requires cutoff near bdy. \( z_0 = \epsilon \).

We will obtain them from 3-pt. \( \langle J_i(z)O_\Delta(x)O^*_\Delta(y) \rangle \):

This alternative works for AdS\(_{d+1}\), but not for RG flows.

i. Conserved current \( J_i(z) \) in CFT\(_d\) has \( \Delta = d - 1 \) and transforms under inversion as

\[
J_i(z) \rightarrow J'_i(z) = |z''^{2(d-1)}| l_{ij}(z') J_j(z')
\]

\( \exists \) unique conformal tensor with correct trf. properties:

\[
\langle J_i(z)O_\Delta(x)O_\Delta(y) \rangle = -\frac{1}{|x-y|^{2\Delta-d+2}|x-z|^{d-2}|y-z|^{d-2}} \left[ \frac{(x-z)_i}{|x-z|^2} - \frac{(y-z)_i}{|y-z|^2} \right] \xi
\]

Find 2-pt. correlator from Ward identity

\[
\frac{\partial}{\partial z_i} \langle J_i(z)O_\Delta(x)O^*_\Delta(y) \rangle = [\delta(x - z) - \delta(y - z)] \frac{2\pi^{d/2} \xi}{\Gamma(d/2)} \frac{1}{|x-y|^{2\Delta}}
\]

\[
= [\delta(x - z) - \delta(y - z)] \langle O_\Delta(x)O^*_\Delta(y) \rangle.
\]
ii. Gravity dual of $\langle J_i \mathcal{O}_\Delta \mathcal{O}_\Delta^* \rangle$:

Bulk dual of $J_i(z)$ is gauge field $A_\mu(w)$

Consider bulk action

$$S = \int \frac{d^{d+1}w}{w_0^{d+1}} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g^{\mu\nu} (\partial_\mu - iA_\mu) \phi (\partial_\nu + iA_\nu) \phi^* + m^2 \phi \phi^* \right]$$
Need a new ingredient, the bulk-bdy prop of gauge field:

\[ K_{\mu i}(w, \vec{z}) = C^d \frac{w_0^{d-2}}{[w_0^2 + (w - \vec{z})^2]^{d-1}} I_{\mu i}(w - \vec{z}) \quad C^d = \frac{\Gamma(d)}{2\pi^{d/2}\Gamma(d/2)} \]

A soltn. of bulk Maxwell eqtn, normalized to \( \delta(\vec{w} - \vec{z}) \) at bdy, with correct inversion symmetry:

\[ K_{\mu i}(w, \vec{z}) \rightarrow w'^2 I_{\mu \nu}(w') K_{\nu j}(w', \vec{z}') I_{ji}(\vec{z}') |\vec{z}'|^{2(d-1)} \]

Transforms as a vector field at bulk point \( w \) and as a (conserved) current at bdy point \( \vec{z} \).

\( K_{\mu i} \) is not unique, one can add pure gauge to obtain

\[ K_{\mu i}(w, \vec{z}) + \frac{\partial}{\partial w_\mu} V_i(w, \vec{z}). \]
With these ingredients, the 3-pt. function is given by the integral

\[ \langle J_i(z)O_\Delta(x)O^*_\Delta(y) \rangle = \int \frac{d^{d+1}w}{w_0^{d-1}} K_{\mu i}(w, z) K_\Delta(w, x) \frac{\partial}{\partial w_\mu} K_\Delta(w, y) \]

Integral can be done using translation symmetry and inversion as before. Result agrees with previous bdy conformal tensor with a specific value of the coefficient \( \xi \). From the Ward identity, we identify

\[ \langle O_\Delta(x)O^*_\Delta(y) \rangle = \frac{(2\Delta - d)\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \frac{1}{|x - y|^{2\Delta}}. \]

Coefficient agrees with the \( \epsilon \) cutoff method, but differs by factor \( (2\Delta - d)/\Delta \) from naive calculation with no cutoff.
C. What good is all this information? Very useful application by Lee, Minwalla, Rangamani, Seiberg, \cite{Lee9806074}.

Prime example of AdS/CFT:

Type IIB SG on $\text{AdS}_5 \otimes S_5 \leftrightarrow \mathcal{N} = 4 \text{ SYM}$.

i. The CFT has chiral primary operators $O_k = \text{Tr}(X^k)$ in irreps $(0, k, 0)$ of $\text{SU}(4)_R$, $k \geq 2$. Lowest states of short irreps of $\text{SU}(2,2|4)$. Thus they have fixed $\Delta = k$. (no anom. dim.)

Normalize 2-pt. functions, so that $\langle O_k(\vec{x})O_l(\vec{y}) \rangle = \frac{\delta_{kl}}{|\vec{x} - \vec{y}|^{2k}}$. 3-pt functions are then given by:

$$\langle O_k(\vec{x})O_l(\vec{y})O_m(\vec{z}) \rangle = \frac{C_{klm}(N, \lambda = g^2 N)}{|\vec{x} - \vec{y}|^{k+l-m}|\vec{y} - \vec{z}|^{l+m-k}|\vec{z} - \vec{x}|^{m+k-l}}$$

Spacetime dependence fixed by conformal sym., but $C_{klm}(N, \lambda = g^2 N)$ expected to depend on $N, \lambda$. 

Dan Freedman
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Early Results on Correlation Functions

14/22
ii. \( \exists \) 1:1 match of the \( O_k \) with bulk scalars \( \phi_k \) in KK spectrum of IIB SG on \( \text{AdS}_5 \otimes S_5 \). \( \phi_k \) \( \in \) irrep \( (0, k, 0) \) and has mass
\[
m_k^2 = k(k - 4) = \Delta(\Delta - 4) \quad \implies \quad \Delta = k.
\]
Kim, Romans, van Nieuwenhuizen, 1985

iii. In a tour de force LMRS obtained the quadratic and cubic couplings of the \( \phi_k \) by careful dimensional reduction of IIB SG.
With these couplings + our results for generic \( \langle O O \rangle \) and \( \langle O O O \rangle \), they find (for normalized 2-pt. functions) that
\[
\lim_{N, \lambda \gg 1} C_{klm}(N, \lambda) = \lim_{N \to \infty} C_{klm}(N, \lambda = 0). \quad \text{The large } N
\]
3-pt. functions are those of the free \( \mathcal{N} = 4 \) theory!
Suspicion that 3-pt. functions are indep. of $\lambda = g^2 N$ for all $N$.
Verified through order $g^2$ and all $N$ by weak coupling calcs. of both 2- and 3-pt.functions in $\mathcal{N} = 4$ SYM:

D’Hoker, DZF, Skiba, 9807098.

Proven by an intricate argument involving ”bonus” $U(1)_Y$ symmetry:

Intriligator, 9811047

Conclusion: This non-renormalization property was a significant new result about $\mathcal{N} = 4$ SYM obtained from AdS/CFT.
D. 4-pt. functions require both exchange and contact Witten diagrams. In recent work Rastelli and Zhou, 1608.06624, 1710.05923 found a compact, exact form for the 4-pt. correlators of the chiral primaries $O_k$ which includes all contributing diagrams. 0-th step in their new work is the fact any exchange diagram can be expressed as a finite sum of contact diagrams.

\[ \sum_{\Delta, \ell} \left( \begin{array}{c} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \\ \vec{x}_4 \end{array} \right) \]

\[ \Delta, \ell \]

\[ \Delta, \ell \]

D'Hoker, DZF, Rastelli 9905049

Let’s show this for the simplest case of exchange of a scalar $O_{k'}$ with 4 identical external $O_k$. (To simplify, all inessential normalization factors and coupling constants are omitted.)
Ingredients:

Bulk Green’s function: \( m^2 = k'(k' - 4) \quad u = \frac{(w - z)^2}{2w_0 z_0} \)

\((-\Box + m^2)G_{k'} = \delta(w, z) \quad G_{k'}(u) = u^{-k'}_2 F_1(, ; ; -2/u)\)

Bulk-bdy. prop. \( K_k(z, \vec{x}) = \left(\frac{z_0}{(z - \vec{x})^2}\right)^k \)

Step i. The scalar exchange integral is

\[ S(\vec{x}_i) = \int dw K_k(w, \vec{x}_3)K_k(w, \vec{x}_4) \int dz G_{k'}(u)K_k(z, \vec{x}_1)K_k(z, \vec{x}_2) \]

Translate all \( \vec{x}_i \rightarrow \vec{x}_i - \vec{x}_1 \) and separate off the z-integral

\[ A(w, \vec{x}_{21}) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_{k'}(u)K_k(z, \vec{0})K_k(z, \vec{x}_{21}) \]

Let \( \vec{v} = \vec{x}_{21} \) and invert \( z, w, \vec{v} \) so that

\[ K_k(z', \vec{0}) \rightarrow z_0^{-k}, \quad K_k(z', \vec{v}') \rightarrow |\vec{v}'|^{2k}\left(\frac{z_0}{(z' - \vec{v}')^2}\right)^k. \]
Integral $\rightarrow \quad A(w, \vec{v}) = |\vec{v}'|^2 k^2 I(w_0', \vec{w}' - \vec{v}')$ with

$$I(w_0, \vec{w}) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_k'(\frac{(w-z)^2}{2w_0z_0}) z_0^k \left( \frac{z_0}{z_0^2 + z^2} \right)^k$$

Observe scale invariance: $I(w_0, \vec{w}) = I(\lambda w_0, \lambda \vec{w})$

**Step ii.** Apply $(-\Box_w + m^2)$ inside the integral

$$0 \leq t \leq 1$$

$$(-\Box_w + m^2) I(w_0, \vec{w}) = \left( \frac{w_0^2}{w_0^2 + \vec{w}^2} \right)^k \equiv t^k \leftarrow \text{scale inv. variable } t.$$ Define $I(w_0, \vec{w}) = f(t)$. From $(-\Box_w + m^2) f(t)$, we get

$$4t^2(t-1)f'' + 4t(t+d/2-1)f' + m^2 f = t^k.$$ Must solve this inhom. 2nd order ODE subject to conditions:

a) $I(w_0, \vec{w})$ is regular at $\vec{w} = 0 \implies f(t)$ is regular at $t = 1$.

b) $G_k' \sim w_0^{k'}$ as $w_0 \rightarrow 0 \implies f(t) \sim t^{k'/2}$ as $t \rightarrow 0$. 

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\[4t^2(t - 1)f'' + 4t(t + d/2 - 1)f' + m^2 f = t^k \quad **.\]

**Step iii.** Assume series form \(\sum_j a_j t^j\). Regularity at \(t = 1\) suggests termination above at \(j = j_{\text{max}}\). Inspect ** to find \(j_{\text{max}} = k - 1\).

Substitute series in ** and get downward recursion relation which terminates below at \(j_{\text{min}} = k'/2\) as needed by condition b).

The number of terms in the series must be a +ve integer which gives \(2(k - 1) - k' = 2n \geq 0\). Thus we find solution:

\[f(t) = \sum_{j=k'/2}^{k-1} a_j t^j\]

\[a_{k-1} = \frac{1}{4(k-1)^2} \quad a_{j-1} = \frac{(j-k'/2)(j+(k'-d)/2)}{(j-1)^2} a_j \quad \text{n.b. } d = 4\]

**Step iv.** Soltn is unique. Homogeneous ODE is hypergeometric eqtn with two soltns which violate conds. a) or b).
Step v. Repristinate original variables $w, \vec{x}_1, \vec{x}_2$ and assemble solution.

Note that $t^j = (w'_0)^j \left( \frac{w'_0}{w'_0^2 + \vec{w}'^2} \right)^j \rightarrow K_j(w, \vec{x}_1) K_j(w, \vec{x}_2)$

The 4-pt. exchange integral becomes:

$$S(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \sum_{j=k'/2}^{k-1} a_j |\vec{x}_{21}|^{2(j-k)} D_{jjkk}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$$

The $D$-functions are 4-pt. contact diagrams:

$$D_{jjkk}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \int \frac{d^5w}{w_0^5} K_j(w, \vec{x}_1) K_j(w, \vec{x}_2) K_k(w, \vec{x}_3) K_k(w, \vec{x}_4)$$

Thus, we have expressed a scalar exchange diagram as a finite sum of contact diagrams.
\[ S(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \sum_{j=k'/2}^{k-1} a_j |\vec{x}_1 - \vec{x}_2|^{2(j-k)} D_{jjkk}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) \]

**Step vi.** Compare this result with the constraint due to SU(4)_R symmetry of \( \mathcal{N} = 4 \) SYM. The cubic couplings \( g_{kkk'} \) of the bulk theory are (restricted to irreps in the direct product):

\[(0, k, 0) \otimes (0, k, 0) = \bigoplus_{j=1}^{k} (0, 2j, 0) + \ldots,\]

where only irreps of the chiral primaries \( O_j \) are included.

One notices that the top operator \( O_{k'=2k} \) is allowed by group theory but is excluded in the diagram above. The reason is that \( \langle O_k O_k O_{2k} \rangle \) is a special case called an "extremal" 3-point function. The corresponding bulk cubic vertex involve derivatives of the fields. With more work, one can show that the diagram with \( \phi_{2k} \) exchanged is also a finite sum of contact diagrams.