Small deformations of circular Wilson loops at weak and strong coupling

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20 Years Later: The Many Faces of AdS/CFT
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**Setup**

- Consider a Wilson loop in $\mathcal{N} = 4$ SYM following a path in $\mathbb{R}^2$ given by
  \[ X(\theta) = x^1(\theta) + ix^2(\theta) = e^{i\theta + g(\theta)}. \]

- We will study this loop in an expansion in small $g$.

- It is convenient to write it in a Fourier decomposition
  \[ g(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta} \]
  and without loss of generality $g(\theta)$ is real, so $b_{-n} = \overline{b}_n$.

- The expectation value of the Wilson loop can be written in an expansion in powers of $b_n$
  \[ \langle W_X \rangle = \langle W \rangle_0 + \langle W \rangle_2 + \langle W \rangle_4 + \cdots, \quad \langle W \rangle_{2n} \sim \mathcal{O}(b^{2n}) \]

- As I will review, the order $b^0$ and $b^2$ are known to all orders in the coupling. What I focus on is $\langle W \rangle_4$. 

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Outline

- Deformed circle and defect CFT.
- Results for \( \langle W \rangle_4 \)
- Construction of string solution for near circular Wilson loop.
- Spectral parameter (in)dependence.
- Conclusions.
Deformed circle and defect CFT

• To lowest order we just have the circular Wilson loop

\[ \langle W \rangle_0 = \frac{1}{N} L_{N-1}^1(\lambda/4N)e^{\lambda/N} \sim \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \]

• Deformations away from the circle can be represented by insertions of adjoint valued fields into the Wilson loop. Normally the first insertion is \( F_{\mu\nu} \dot{x}^\nu \).

• For a radial deformation of the circular Maldacena-Wilson loop this is replaced with

\[ F_{r\phi} = F_{r\phi} + iD_r \Phi^1 \]

• The circular Wilson loop preserves an \( SL(2, \mathbb{R}) \) subgroup of the conformal group (also \( SO(3) \times SO(5) \), and their supergroup completion).

• All insertions can be classified by representations of this group, and this first insertion, the displacement operator is in fact a protected operator of dimension 2.
At order $g^2$ the expectation value of the Wilson loop is therefore given by

$$\langle W \rangle_2 = \int g(\theta_1)g(\theta_2)\langle \langle F(\theta_1)F(\theta_2) \rangle \rangle d\theta_1 d\theta_2$$

$$= B(\lambda) \int \frac{\bar{g}(\theta_1)g(\theta_2)}{16 \sin^4 \frac{\theta_1 - \theta_2}{2}} d\theta_1 d\theta_2$$

$$= 8\pi^2 B(\lambda) \sum_{n=2}^{\infty} n(n^2 - 1)|b_n|^2,$$

The factor of $B(\lambda)$ is related to the normalization of the displacement operator, and by studying deformations that preserve supersymmetry can also be fixed from localization

$$B(\lambda) = \frac{1}{4\pi^2} \frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}.$$ 

The correlator of insertions is

$$\langle \langle F(\theta_1)F(\theta_2) \rangle \rangle = \text{Tr} \mathcal{P} \left[ e^{\int_{\theta_1}^{\theta_1} (iA + \Phi)} F(\theta_1) e^{\int_{\theta_2}^{\theta_2} (iA + \Phi)} F(\theta_2) e^{\int_{2\pi}^{2\pi} (iA + \Phi)} \right]$$
• At one loop in perturbation theory, the VEV of the Wilson loop is
\[ \langle W \rangle = \frac{\lambda}{16\pi^2} \text{Tr} \mathcal{P} \int \frac{-\dot{X}(\theta_1)\dot{X}(\theta_2) - \dot{X}(\theta_1)\dot{X}(\theta_2) + 2|\dot{X}(\theta_1)\dot{X}(\theta_2)|}{2|X(\theta_1) - X(\theta_2)|^2} d\theta_1 d\theta_2 + \cdots \]

• We can write this as an expansion in powers of \( g(\theta_1) \) or alternatively \( b_n \).

• I stated the results for order \( b^0 \) and \( b^2 \) above. We also evaluated the \( b^4 \) term.

• From the point of view of the defect-CFT, we need to consider insertions of operators of higher dimension and higher point functions, so we can write the Wilson loop as (schematically)
\[ \langle W \rangle = \langle W \rangle_0 + \int g(\theta_1)g(\theta_2)\langle [F(\theta_1)F(\theta_2)] \rangle d\theta_1 d\theta_2 \\
+ \int g^2(\theta_1)g^2(\theta_2)\langle [DF(\theta_1)DF(\theta_2)] \rangle d\theta_1 d\theta_2 \\
+ \int g^2(\theta_1)g(\theta_2)g(\theta_3)\langle [DF(\theta_1)F(\theta_2)F(\theta_3)] \rangle d\theta_1 d\theta_2 d\theta_3 \\
+ \int g(\theta_1)g(\theta_2)g(\theta_3)g(\theta_4)\langle [F(\theta_1)F(\theta_2)F(\theta_3)F(\theta_4)] \rangle d\theta_1 d\theta_2 d\theta_3 d\theta_4 + \cdots \]

• The one-loop graph above sees only 2-point functions.
• At two loops order the graphs combine to the elegant expression
\[
\langle W[C] \rangle_{2\text{-loop}} = -\frac{\lambda^2}{128\pi^4} \int \epsilon(\theta_1, \theta_2, \theta_3) I(\theta_1, \theta_3) \frac{x_{32} \cdot \dot{x}_2}{x_{32}^2} \log \frac{x_{21}^2}{x_{31}^2} d\theta_1 d\theta_2 d\theta_3 \\
+ \frac{\lambda^2}{2} \left( \frac{1}{16\pi^2} \int I(\theta_1, \theta_2) d\theta_1 d\theta_2 \right)^2 - \frac{\lambda^2}{64\pi^4} \int_{\theta_1 > \theta_2 > \theta_3 > \theta_4} I(\theta_1, \theta_3) I(\theta_2, \theta_4) d\theta_1 d\theta_2 d\theta_3 d\theta_4,
\]
\[
I(\theta_1, \theta_2) = \frac{\dot{x}_1 \cdot \dot{x}_2 + |\dot{x}_1||\dot{x}_2|}{x_{12}^2}.
\]
• This determines some one-loop anomalous dimensions, some structure constants and the tree-level 4-point function.
• Using this comparison we were able to find the normalizations of some operators, some anomalous dimensions and some structure constants.
Results to order $b^4$

- We expand the integrand of the one-loop diagram to order $b^4$ and integrate. The result is

$$\langle W \rangle_4 \sim \lambda \sum_{n=1}^{\infty} \left( \frac{49n^5 - 10n^3 - 3n}{96} |b_n|^2 + n^5 b_n^3 \bar{b}_n^3 \right)$$

$$+ \lambda \sum_{n \neq m} \left[ \left( - \frac{1}{24}(n^2 - 1)|n|(m^2 + 4n^2 - 3) + \frac{1}{48}(4m^4 - 14m^3n + 23m^2n^2 - 7m^2 - 14mn^3 + 12mn + 4n^4 - 7n^2 + 3)|n - m| \right) |b_n b_m|^2 
\right.$$  

$$+ \left( \frac{3}{8}(3n^2 - 1)|n|^3 - \frac{1}{96}(m^2 - 1)(3n^2 + 2mn + 4m^2 - 3)|m| 
- \frac{1}{96}((2n + m)^2 - 1)(15n^2 + 14mn + 4m^2 - 3)|2n + m| + \frac{1}{48}(6n^4 + 14mn^3 + 23m^2n^2 - 9n^2 + 16m^3n - 14mn + 4m^4 - 7m^2 + 3)|n + m| \right) b_n^2 b_m \bar{b}_{2n+m}$$

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\[-\frac{\lambda}{48} \sum_{n < m < p} \left( (n^2 - 1) |n|(m^2 + mn + mp + 4n^2 + np + p^2 - 3) \\
+ \left((m + n + p)^2 - 1\right) |m + n + p|(4m^2 + 7mn + 7mp + 4n^2 + 7np + 4p^2 - 3) \\
- |m + n|(4m^4 + 15m^3n + m^3p + 22m^2n^2 + m^2p^2 - 7m^2 + 15mn^3 - mnp^2 \\
- 13mn - mp + 4n^4 + n^3p + n^2p^2 - 7n^2 - np - p^2 + 3) \right) b_n b_m b_p \bar{b}_{n+m+p}\]

- We also have the full results for two loop order.
- It is much more complicated, and we don’t have it in closed form, but rather a very efficient algorithm to calculate it for arbitrary $n$, $m$, $p$. 
AdS calculation

• The holographic dual of a Wilson loop in $\mathbb{R}^2$ is a fundamental string in AdS$_3$.

• Using the Pohlmeyer reduction, one ends with the generalized cosh-Gordon equation

$$\partial \bar{\partial} \alpha = e^{2\alpha} + |f|^2 e^{-2\alpha}$$

where $f$ is an arbitrary holomorphic function.

• Given $\alpha$ and $f$, the regularized action is given by the integral over the disc

$$A_{\text{reg}} = -2\pi - 2 \int_{\Sigma} dz d\bar{z} |f(z)|^2 e^{-2\alpha(z, \bar{z})},$$

• The shape of the Wilson loop is obscured in this description...

• For the circle $f = 0$ and $\alpha = -\ln(1 - z\bar{z})$, so for nearly circular Wilson loops we can take small $f$

$$f(z) = \epsilon \sum_{p=0}^{\infty} a_p z^p.$$ 

and solve for $\alpha$ perturbatively.
• There is also a prescription to derive the shape of the Wilson loop perturbatively in $\epsilon$.

• Expressing the result in terms of $b_n$, at linear order in $\epsilon$ we find

$$b_n = \begin{cases} 
0, & n = -1, 0, 1 \\
\frac{2\epsilon a_{n-2}e^{i\varphi}}{n(n^2 - 1)}, & n \geq 2 \\
\frac{-2\epsilon \bar{a}_{n-2}e^{-i\varphi}}{n(n^2 - 1)}, & n \leq -2 
\end{cases}$$

• Note that there is an extra spectral parameter $\varphi$ in the mapping.

• Given $f$ and $\alpha$ there is a one-parameter family of different string solutions, (and Wilson loops). All have the same area, so the same VEV for the Wilson loop at strong coupling.

• Dekel studied many examples of curves and found that at weak coupling there is a dependence on $\varphi$, but only at order $\epsilon^8$ or higher.

• One of the reasons for our examination was to verify whether this is correct and to try to understand why. What is the analog of the spectral parameter at weak coupling?
To the next order in $\epsilon$

\[
g(\theta) = \epsilon \sum_{p=2}^{\infty} \left( \frac{2a_{p-2}e^{i(p\theta+\varphi)}}{p(p^2 - 1)} + \frac{2\bar{a}_{p-2}e^{-i(p\theta+\varphi)}}{p(p^2 - 1)} \right)
\]

\[
+ \epsilon^2 \sum_{p=0}^{\infty} \left[ \frac{a_{p-2}^2(5p^2 + 1)e^{2i(p\theta+\varphi)}}{p(p^2 - 1)^2(4p^2 - 1)} - \frac{4|a_{p-2}|^2}{p^2(p^2 - 1)^2} - \frac{\bar{a}_{p-2}^2(5p^2 + 1)e^{-2i(p\theta+\varphi)}}{p(p^2 - 1)^2(4p^2 - 1)} \right]
\]

\[
+ \sum_{p>q} \left[ \frac{4a_{p-2}a_{q-2}(p^2 + 3pq + q^2 + 1)e^{i((p+q)\theta+2\varphi)}}{(p-1)(p+1)(q-1)(q+1)(p+q-1)(p+q)(p+q+1)} \right.
\]

\[
- \frac{4\bar{a}_{p-2}\bar{a}_{q-2}(p^2 + 3pq + q^2 + 1)e^{-i((p+q)\theta+2\varphi)}}{(p-1)(p+1)(q-1)(q+1)(p+q-1)(p+q)(p+q+1)}
\]

\[
- \frac{4a_{p-2}\bar{a}_{q-2}e^{i(p-q)\theta}}{p(p^2 - 1)(q-1)(q+1)} + \frac{4\bar{a}_{p-2}a_{q-2}e^{-i(p-q)\theta}}{p(p^2 - 1)(q-1)(q+1)} \right]
\]

\[+ O(\epsilon^3).\]

We calculate also the $O(\epsilon^3)$ term.
• We can plug the $b_n(\epsilon, a_p)$ into the expression we found for the one loop VEV of the Wilson loop.

• The result is

\[
\lambda \epsilon^2 \sum_{p=2}^{\infty} \frac{4}{p(p^2 - 1)} |a_{p-2}|^2 - \lambda \epsilon^4 \sum_{p=2}^{\infty} \frac{8(17p^4 + 4p^2 + 3)}{3p^3(p^2 - 1)^3(4p^2 - 1)} |a_{p-2}|^4
\]

\[
- \lambda \epsilon^4 \sum_{p > q \geq 2} \frac{32|a_{p-2}a_{q-2}|^2}{3p^2(p^2 - 1)^2q(q^2 - 1)(p + q)((p + q)^2 - 1)}
\]

\[
(p^5 + 9p^4q + 25p^3q^2 + 7p^2q^3 - 6pq^4 - 2q^5 - 4p^3 + 8p^2q + 5pq^2 - q^3 + 3p + 3q)
\]

\[
+ 16(72p^4 - 71p^3q + 16p^2q^2 - 3p^2 - 4q^2 + 11pq + 3)(a_{p-2}^2\tilde{a}_{q-2} - \tilde{a}_{2p-q-2} + a_{q-2}\tilde{a}_{p-2}^2a_{2p-q-2})
\]

\[
- \lambda \epsilon^4 \sum_{p > q > r \geq 2} \frac{32(a_{p-2}a_{q-2}\tilde{a}_{r-2}\tilde{a}_{p+q-r-2} + \tilde{a}_{p-2}a_{q-2}a_{2r-2}a_{2p+q-r-2})}{3p(p^2 - 1)q(q^2 - 1)(p + q)((p + q)^2 - 1)((p + q - r)^2 - 1)(p + q - r)}
\]

\[
(p^5 + p^4(16q - 7r) + p^3(55q^2 - 34qr + 4r^2 - 4) + p^2(55q^3 - 60q^2r + 12qr^2 + q + 7r)
\]

\[
+ p(16q^4 - 34q^3r + q^2(12r^2 + 1) + 8qr - 4r^2 + 3) + q(q^2 - 1)(q^2 - 7qr + 4r^2 - 3))
\]

• There is no dependence on $\varphi$ at this order in $\lambda$ and $\epsilon$.

• The same is true at order $\lambda^2$.

• We found some examples with $\varphi$ dependence at $\mathcal{O}(\epsilon^6)$. 
Summary

- We are calculating the expectation value of deformed circular Wilson loops to the next order.
- Explicit results for one-loop, two-loop and soon strong coupling.
- Explicit results for the dimensions and structure constants of insertions into the Wilson loop.
- A surprising independence on the spectral parameter $\varphi$ at order $\epsilon^4$.
- Are there any other hidden structures in our results?
Summary

- We are calculating the expectation value of deformed circular Wilson loops to the next order.
- Explicit results for one-loop, two-loop and soon strong coupling.
- Explicit results for the dimensions and structure constants of insertions into the Wilson loop.
- A surprising independence on the spectral parameter $\varphi$ at order $\epsilon^4$.
- Are there any other hidden structures in our results?
- This calculation could have been done almost 19 years ago.
- We still find surprises in classical $AdS$/CFT calculations.
The end