$p$-adic AdS/CFT

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1. Reminiscences

In 1995-1997, I thought the most urgent problems related to highly dynamical branes.

- String dualities seemed to rely excessively on BPS sectors.
- DBI actions for branes were known. I wondered, could we pull some trick like
  \[ \text{Nambu-Goto} \rightarrow \text{Polyakov} \]

But highly dynamical branes were too hard!
(de Wit-Hoppe-Nicolai ’88 was depressing; BFSS was conceptually deep but hard to calculate with.)

Witten’s work on non-abelian gauge theories on multiple branes was impressive, but many branes together seemed very non-generic to me.

- I was more impressed by Horava-Witten’s $E_8$ end-of-the-world brane.
I felt better about multiple branes together after the work with Igor Klebanov and Amanda Peet (also Strominger unpublished) on near-extremal D3-brane entropy: $S_{\text{brane}} = (3/4)^{1/4} S_{\text{free}}$ if we fix $E$.

Perfect agreement for dilaton scattering (Klebanov) and for gravitons (from protected $\langle TT \rangle$ correlator) were very striking and put us in the right frame of mind...

I remember Curt Callan asking me, “So, what closed string calculation is going to tell you the three-point function in the gauge theory?”
Juan Maldacena’s landmark 1997 paper unified a lot of strands of thought and identified the key point: A whole geometry could be precisely dual to the renormalizable limit of the action on multiple branes.

- But Curt’s question didn’t really seem to be answered.

Since the problem was now field theory after all (and not DBI as I had expected), the burning question in discussions with Klebanov and Polyakov was how to get at the generating functional of connected Green’s functions.

- $S_{\text{supergravity}} = W_{\text{connected correlators}}$ eventually seemed most natural—with $S_{\text{supergravity}}$ corrected by loops $(1/N^2)$ to $\Gamma_{\text{effective}}$, and by stringiness $(1/g_{YM}^2 N)$.

- Imposing boundary conditions at boundary of AdS (or a cutoff between AdS and flat space) was well motivated by all those matching calculations.
2. And now for something completely different...

Consider the furthest neighbor Ising model:

After using the digit-reversing Monna map, far apart spins are close and vice versa. The $2$-adic norm formalizes this alternative notion of closeness: $|i - j|_2 = 2^{-d(i, j)/2}$ where $d(i, j)$ is distance on the tree.
Dyson ’69: Furthest neighbor Ising model has a finite temperature phase transition.

Missarov-Lerner ’89 (also Bleher-Sinai ’75 and others) showed that the critical theory at the transition is characterized by $\phi^4$ theory over the “2-adic numbers $\mathbb{Q}_2$:

$$S = -\int_{\mathbb{Q}_2 \times \mathbb{Q}_2} dx \, dy \frac{1}{2} \frac{\phi(x) \phi(y)}{|x - y|^2} + \int_{\mathbb{Q}_2} dx \left[ \frac{r}{2} \phi^2 + \frac{\lambda^4}{4!} \phi^4 \right]$$

where $\phi : \mathbb{Q}_2 \to \mathbb{R}$

and $s$ is a parameter.

Ordinary conformal invariance is not realized, but $PGL(2, \mathbb{Q}_2)$ is.

- $z \to \frac{az+b}{cz+d}$ with $a, b, c, d, z \in \mathbb{Q}_2$.

- These LFTs map spin clusters to spin clusters, but sometimes changing the size of the clusters.

- $\langle O(z)O(0) \rangle \propto \frac{1}{|z|^{2\Delta}}$, similar to usual CFTs.
Slight generalization: If \( x = p^{v/a} b \) with \( p \nmid a, b \) then \( |x|_p \equiv p^{-v} \). And \( |0|_p = 0 \).

*The p-adic numbers* \( \mathbb{Q}_p \) *are the completion of* \( \mathbb{Q} \) *wrt* \( | \cdot |_p \).

\( \mathbb{R} \) *is the completion of* \( \mathbb{Q} \) *wrt the absolute value on* \( \mathbb{Q} \), *which we denote* \( | \cdot |_\infty \).

- **Intuition:** \( p \) *is small but non-zero in* \( \mathbb{Q}_p \).
- **Intuition:** The completion \( \mathbb{Z}_p \) of \( \mathbb{Z} \) *is similar to* \([-1, 1]\) *because* \( x \in \mathbb{Z}_p \) *iff* \( |x|_p \leq 1 \).
- **Any** \( x \in \mathbb{Q}_p \) *can be written* \( x = \sum_{m=-v(x)}^\infty a_m p^m \) *with* \( a_m \in \{0, 1, \ldots, p-1\} \).
- **Surprising relation** *(the start of adeles)*:

\[
\prod_v |x|_v \equiv |x|_\infty \prod_p |x|_p = 1 \quad \text{for} \quad x \in \mathbb{Q}.
\]

(1)

Example: \( |15|_\infty = 15 \), while \( |15|_3 = \frac{1}{3} \) and \( |15|_5 = \frac{1}{5} \).

- **Defining** \( \zeta_p(s) \equiv \frac{1}{1-p^{-s}} \) *and* \( \zeta_\infty(s) \equiv \pi^{-s/2} \Gamma_{\text{Euler}}(s/2) \), *find\n
\[
\zeta_{\text{Riemann}}(s) = \prod_p \zeta_p(s), \quad \zeta_A(s) \equiv \prod_v \zeta_v(s) \overset{!}{=} \zeta_A(1 - s)
\]

(2)
3. \textit{p-adic AdS/CFT}

\( \mathbb{Q}_p \cup \{ \infty \} = P^1(\mathbb{Q}_p) \) is the boundary of the Bruhat-Tits tree:

\[
T_p = \text{PGL}(2, \mathbb{Q}_p)/\text{PGL}(2, \mathbb{Z}_p) \quad \text{like} \quad \mathbb{H} = E\text{AdS}_2 = \text{SL}(2, \mathbb{R})/U(1).
\]

- Choosing a \( p \)-adic number \( z \in \mathbb{Q}_p \) amounts to choosing a path up through the tree \( T_p \).
- Each node \( a \) on the way up to \( z \) is a rational approximation to \( z \), so that \( |z - a|_p \leq |z_0|_p \).
- We can label a node by \( (z_0, z) \) where \( z \in \mathbb{Q}_p \).
- \( z \in \mathbb{U}_p \) means \( |z|_p = 1 \), so \( \mathbb{U}_p \) is like the unit circle.
What’s an AdS/CFT practitioner to do once it’s clear that $T_p$ is like $E \text{AdS}$?

[Some precursors to our work: Freund-Olson-Witten ’89, Zabrodin ’88, Manin-Marcolli ’02, Harlow-Shenker-Stanford-Susskind ’11; see also Heydeman-Marcolli-Saberi-Stoica ’16]

**Define** a CFT on boundary from simple dynamics on $T_p$, e.g.

$$S_{\text{bulk}} = \sum_{\langle ab \rangle} \frac{1}{2} (\phi_a - \phi_b)^2 + \sum_a \left( \frac{1}{2} m^2 \phi_a^2 + \frac{g_3}{3!} \phi_a^3 + \frac{g_4}{4!} \phi_a^4 \right)$$  \hspace{1cm} (3)

in lieu of $S_{\text{supergravity}}$. Immediately obtain

$$G_{\phi\phi}(a, b) = \frac{\zeta_p(2\Delta)}{p^\Delta} p^{-\Delta d(a,b)} \quad \text{where} \quad m^2 = -\frac{1}{\zeta_p(\Delta - 1)\zeta_p(-\Delta)}$$

$$ \geq m^2_{\text{BF}} \equiv -\frac{1}{\zeta_p(-1/2)^2}$$  \hspace{1cm} (4)

![Diagram](image)

Bulk point $(z_0, z) \in T_p$  \hspace{2cm} Boundary point $x \in \mathbb{Q}_p$

$$K^{p\text{-adic}}_{\phi}(z_0, z; x) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - 1)} \frac{|z_0|^\Delta}{|(z_0, z - x)|_s^{2\Delta}} \quad \text{where} \quad |(X, Y)|_s \equiv \max\{|X|, |Y|\}$$

$$K^{\text{real}}_{\phi}(z_0, \bar{z}; \bar{x}) = \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta - n)} \frac{z_0^\Delta}{(z_0^2 + (\bar{z} - \bar{x})^2)^\Delta}, \quad \text{Note simple form of standard} \quad \text{AdS}_{n+1}/\text{CFT}_n \quad \text{prefactor!}$$
The \( p \)-adic version of the three-point calculation factorizes conveniently into external legs times an internal summation (easier with geodesic bulk diagrams):

\[
\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3) \rangle = -g_3 \sum_x \prod_{i=1}^3 K(x; z_i)
\]

\[
= -g_3 \left[ \prod_{i=1}^3 K(c; z_i) \right] \times \sum_x \hat{G}(c, b) \hat{G}(b, x)^3
\]

\[
= \frac{C^{(p)}_{\mathcal{O}\mathcal{O}\mathcal{O}}}{|z_1 z_2 z_3|^{\Delta}}.
\]

\[
C^{(v)}_{\mathcal{O}\mathcal{O}\mathcal{O}} = -g_3 \frac{\zeta_v(\Delta)^3 \zeta_v(3\Delta - 1)}{\zeta_v(2\Delta - 1)^3}
\]

holds equally for \( v = p \) and \( v = \infty \).

\[
\prod_v \frac{\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3) \rangle_v}{-g_3} \xrightarrow{\text{adelic}} \frac{\zeta_A(\Delta)^3 \zeta_A(3\Delta - 1)}{2\zeta_A(2\Delta - 1)^3}
\]

for \( z_i \in \mathbb{Q} \).  \( \text{(5)} \)
4. \( p \)-adic melonic field theories

Critical behavior in the neighbor Ising model lead to theories of the form

\[
S = \int_{\mathbb{Q}_p} dk \frac{1}{2} \phi(-k)|k|^s \phi(k) + \int_{\mathbb{Q}_p} dx \, V(\phi). \tag{6}
\]

If we’re aiming at something like SYK, then kinetic term should be more like \( \frac{i}{2} \psi \partial_t \psi \).

In frequency space, \( i \partial_t = \omega = |\omega| \text{sgn} \omega \). So try

\[
S_{\text{free}} = \int_K d\omega \frac{1}{2} \psi^{a_1b_1c_1}(-\omega) \Omega_{a_1a_2} \Omega_{b_1b_2} \Omega_{c_1c_2} |\omega|^s (\text{sgn} \omega) \psi^{a_2b_2c_2}(\omega) \tag{7}
\]

where either \( \Omega = 1_{N \times N} \quad (O(N), \quad \sigma_\Omega = +1) \) or

\[
\Omega = \sigma_2 \otimes 1_{N_2 \times N_2} \quad (Sp(N), \quad \sigma_\Omega = -1), \quad \text{and}
\]

\[
S_{\text{int}} = \int_K dt \frac{g}{4} \Omega_{a_1a_2} \Omega_{a_3a_4} \Omega_{b_1b_3} \Omega_{b_2b_4} \Omega_{c_1c_4} \Omega_{c_2c_3} \psi^{a_1b_1c_1}(t) \psi^{a_2b_2c_2}(t) \psi^{a_3b_3c_3}(t) \psi^{a_4b_4c_4}(t).
\]

Statistics of \( \psi \) could be bosonic, \( \sigma_\psi = +1 \),

or fermionic, \( \sigma_\psi = -1 \).

\( K = \mathbb{R} \) or \( \mathbb{Q}_p \).
Problem: \( \text{sgn} \omega \) has several possible meanings on \( \mathbb{Q}_p \).

On \( \mathbb{R} \), \( \text{sgn} \omega = 1 \) iff \( \omega = z\bar{z} = (x + iy)(x - iy) \) for some \( x, y \in \mathbb{R} \). Clearly depends on extending \( \mathbb{R} \) by \( i = \sqrt{-1} \): the unique quadratic extension.

Let \( K = \mathbb{R} \) or \( \mathbb{Q}_p \). To define \( \text{sgn}_\tau \) on \( K \) for any \( \tau \in K \).

- Say \( z = x + \sqrt{\tau}y, \bar{z} = x - \sqrt{\tau}y \): these are numbers in field extension \( K[\sqrt{\tau}] \).
- \( \text{sgn}_\tau \omega = 1 \) iff \( \omega = z\bar{z} \). Else \( \text{sgn}_\tau \omega = -1 \).
- Careful! Sometimes \( \text{sgn}(-1) = +1 \! \)!
- \( \text{sgn}_1 \omega = 1 \) for all \( \omega \).
- For odd \( p \), four choices: \( \tau = 1, p, \epsilon p, \epsilon \) where \( \epsilon^{p-1} = 1 \).
- For \( p = 2 \), eight choices, \( \tau = \pm 1, \pm 2, \pm 3, \pm 6 \).

For \( S_{\text{free}} \) to make sense, must require \( \text{sgn}(-1) = \sigma_\psi \sigma_\Omega \).
Given \( F(t) = \langle \psi(t)\psi(0) \rangle_{\text{free}} \), find \( G(t) = \langle \psi(t)\psi(0) \rangle \) using Schwinger-Dyson:

\[
G(t) = F(t) + \sigma \Omega g^2 N^3 (G \ast G^3 \ast F)(t).
\]

(8)

Convolutions can be handled using a general identity

\[
(\pi \ast \pi')(t) = B(\pi\pi_1, \pi'\pi_1)(\pi\pi'\pi_1)(t).
\]

(9)

- \( \pi_s(t) \equiv |t|^s \).
- \( \pi_{s,\text{sgn}}(t) \equiv |t|^s \text{sgn } t \) is a more general multiplicative character, with “spin.”
- \( B(\pi, \pi') \equiv \frac{\Gamma(\pi)\Gamma(\pi')}{\Gamma(\pi\pi')} \) where \( \Gamma(\pi) \equiv \int_K \frac{dt}{|t|} e^{2\pi i \{t\}} \pi(t) \).

Infrared limit à la Kitaev gives

\[
G(t) = \frac{b \text{sgn}(t)}{|t|^{1/2}} \quad \text{where} \quad \frac{1}{b^4 g^2 N^3} = -\sigma \Omega \Gamma(\pi_{-\frac{1}{2},\text{sgn}})\Gamma(\pi_{\frac{1}{2},\text{sgn}})
\]

(10)
For every field and every choice of $\text{sgn}$, there is exactly one choice of $\sigma_\Omega$ and $\sigma_\psi$ that makes possible the flow \[ \text{(free theory)} \longrightarrow \text{(Kitaev IR)}: \]

<table>
<thead>
<tr>
<th>$K$</th>
<th>condition</th>
<th>$\tau$</th>
<th>$\Gamma(\pi^{-1/2,\text{sgn}})\Gamma(\pi^{1/2,\text{sgn}})$</th>
<th>$\sigma_\Omega$</th>
<th>$\sigma_\psi$</th>
<th>explanation</th>
</tr>
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<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td></td>
<td>1</td>
<td>$-4\pi$</td>
<td>1</td>
<td>1</td>
<td>$O(N)$ bosonic</td>
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<td></td>
<td>$-1$</td>
<td>$-4\pi$</td>
<td>1</td>
<td>$-1$</td>
<td>$O(N)$ fermionic</td>
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<tr>
<td>$\mathbb{C}$</td>
<td></td>
<td>1</td>
<td>$-4\pi^2$</td>
<td>1</td>
<td>1</td>
<td>$O(N)$ bosonic</td>
</tr>
<tr>
<td>$\mathbb{Q}_p$</td>
<td>$p$ odd</td>
<td>1</td>
<td>$-(p + \sqrt{p} + 1)/p^{3/2}$</td>
<td>1</td>
<td>1</td>
<td>$O(N)$ bosonic</td>
</tr>
<tr>
<td>$\mathbb{Q}_p$</td>
<td>$p$ odd</td>
<td>$\epsilon$</td>
<td>$(p - \sqrt{p} + 1)/p^{3/2}$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$Sp(N)$ fermionic</td>
</tr>
<tr>
<td>$\mathbb{Q}_p$</td>
<td>$p \equiv 1 \mod 4$</td>
<td>$p$</td>
<td>$1/p$</td>
<td>$-1$</td>
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</tr>
</tbody>
</table>

For the shaded rows, *entire RG flow is analytically tractable.*
Assuming $\text{sgn } t$ depends not just on $|t|$ but also its “direction” in $\mathbb{Q}_p$, the function $S(t) = \delta_{\nu(t)} \text{sgn } t$ has a local Fourier transform $\hat{S}(\omega) \propto \delta_{\nu(\omega)+1} \text{sgn } \omega$.

Ansatz:

\[
F(t) = \theta p^{\nu(t)/2} f_{\nu(t)} \text{sgn } t \\
G(t) = \theta p^{\nu(t)/2} g_{\nu(t)} \text{sgn } t
\]  \hspace{1cm} (11)

where $\theta^2 = \text{sgn}(-1)$. Wind up with

\[
\frac{1}{f_v} = \frac{1}{g_v} + \frac{\sigma_\psi g^2 N^3}{p} g^3_v.
\]  \hspace{1cm} (12)
5. Conclusions

I hope that \( p \)-adic AdS/CFT is the tip of a big iceberg!

- A great deal of QFT lore must have \( p \)-adic counterparts. See e.g. Melzer ’89.
- Unifying treatment between \( \mathbb{R} \) and \( \mathbb{Q}_p \) brings out universal features of QFT and of AdS/CFT.
- Naturally discrete bulk spacetime \( T_p \) hints at some easier version of quantized “gravity.”
- What is “spin”? Rotation group \( \mathbb{U}_p \) is abelian; obvious local bulk symmetries are finite groups, e.g. \( PGL(2, \mathbb{F}_p) \).
- Relationship to tensor networks is tantalizing.
- How about Lorentzian signature, horizons etc.?
Extra slides
More creative couplings discussed with Monika Schleier-Smith can *interpolate* between Archimedean order (0123...) and hierarchical order (tree-like): e.g. put real “spins” $\phi_i$ at integer points and set

$$H = -\frac{1}{2} \sum_{i,j} J_{i-j} \phi_i \phi_j$$

with

$$J_{\pm 2^n}^{\text{sparse}} = 2^{ns} \quad (s \text{ is a parameter})$$

\[c.f. \quad J_{h}^{\text{p-adic}} = |h|^{-s-1}\]
From \[ H = - \sum_{i,j} J_{ij} \phi_i \phi_j \] get \[ \langle \phi_i \phi_j \rangle \equiv G_{ij} \sim -(J^{-1})_{ij}. \]